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ELEMENTS

OF

CONIC SECTIONS.

COMPILED BY

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It is intended for the use of those persons whose mathematical pursuits are designed to be very limited; but it is believed that it will be found to contain nearly all the properties of the Conic Sections, a knowledge of which is essential to the study of the elements of Physics, Mechanics and Astronomy. In the present edition, a chapter has been added on the curvature of the ellipse and parabola.

The Geometry referred to is Davies' Legendre.

CONIC SECTIONS.

CHAPTER I.

THE ELLIPSE.

DEFINITIONS.

I. THE ELLIPSE is a curve in which the sum of the distances of each point from two fixed points called the *foci*, is equal to a given line.

Let F and F′ [Fig. 1] be the foci of an ellipse ABA′B′: then if from any point M of the curve the straight lines MF and MF′ be drawn, their sum MF+MF′ is equal to a given line.

II. The straight line drawn through the foci, and terminated by the curve, is called the *transverse* or *major* axis. The middle of that part of the transverse axis which lies between the foci, is called the *centre* of the ellipse. The straight line drawn through the centre, at right angles to the transverse axis, and terminated by the curve, is called the *conjugate* or *minor axis.*

Thus, if the straight line joining F and F′ be produced to meet the curve in A and A′, and through C the middle of FF′ the straight line BB′ be drawn perpendicular to AA′, and produced to meet the curve in B and B′; then is C the centre, AA′ the transverse axis, and BB′ the conjugate axis of the ellipse ABA′B′.

III. The distance from the centre to either focus is called the *eccentricity*.

Thus FC, or its equal F′C, is the eccentricity of the ellipse ABA′B′.

IV. A line drawn from any point of the curve, at right angles to either axis, and terminated by it, is called an *ordinate* to that axis. The parts into which the axis is divided by the ordinate, are called *abscissas*.

Thus the perpendicular MP is an ordinate to the transverse axis, and AP, A′P are the corresponding abscissas.

V. A *tangent* is a straight line which has one point in common with the curve, and all its other points without it. That part of either axis produced which is intercepted by a tangent and the ordinate drawn from the point of contact, is called a *subtangent*.

Thus [Fig. 9], if MT be a tangent to the curve at M, PT is the corresponding subtangent.

VI. A straight line drawn perpendicular to a tangent, from the point of contact, and terminated by the axis, is called a *normal*. The part of the axis intercepted by a normal and the corresponding ordinate, is called a *subnormal*.

Thus, MQ, perpendicular to MT, is a normal to the curve at the point M; and PQ is the corresponding subnormal.

VII. The *parameter* is a third proportional to the transverse and conjugate axis.

That is, if the parameter be denoted by p, then

$$AA' : BB' :: BB' : p.$$

VIII. A straight line drawn from any point in the curve, to one of the foci, is called a *radius vector*.

Thus MF is the radius vector of the point M, referred to the focus F.

PROPOSITION I.

PROBLEM.

To describe an ellipse.

Let F and F′ [Fig. 2] be the foci, and X the given line.

FIRST METHOD : BY POINTS.

Draw the indefinite straight line FF′; and from C, the middle of FF′, lay off CA and CA′, each equal to half the given line X : then are A and A′ points of the curve.

For, by construction,

$$CA-CF = CA'-CF'; \text{ that is, } FA = F'A'.$$

Hence, $F'A+FA = F'A+F'A' = AA' = X;$

and also, $FA'+F'A' = FA'+FA = AA' = X:$

consequently A and A′ (Def. I.) are points of the curve.

To find other points : On the line AA′, and between the points F and F′, take any point L, and from F and F′ as centres, and with radii equal respectively to AL and A′L, describe two arcs of circles ; the points M and M′ in which these arcs intersect, are points of the curve.

For, $FM+F'M = AL+A'L = AA' = X;$

and also, $FM'+F'M' = AL+A'L = AA' = X.$

Any number of points of the ellipse may thus be determined.

Corollary I. The sum of the straight lines drawn from any point of the ellipse to the foci, is equal to the transverse axis.

Cor. II. The line MM′ is perpendicular to the transverse axis, and bisected by it. The ellipse is therefore divided by the transverse axis into two equal parts, similarly situated with regard to that axis.

Cor. III. The axes are bisected at the centre.

SECOND METHOD : BY CONTINUOUS MOTION.

Let the extremities of a thread equal in length to the given line, be fixed at the points F and F′, and let a pencil be made to glide along the thread so as to keep it constantly tense ; the curve described by the point M of the pencil is an ellipse. For in every position of M,

$$FM + F'M = X.$$

PROPOSITION II.

THEOREM.

The distance from either extremity of the conjugate axis, to one of the foci, is equal to the semi-transverse axis.

From F and F′ [Fig. 2] draw the straight lines FB and F′B to B, one of the extremities of the conjugate axis ; then is FB = AC.

Since FC = F′C, and BC is common to the two right angled triangles FBC and F′BC, these triangles are equal,

and $FB = F'B$;

and hence, $FB + F'B = 2\,FB$;

but (Prop. I. Cor. I.), $FB + F'B = AA' = 2\,AC$;

therefore, $FB = AC$.

PROPOSITION III.

THEOREM.

The semi-conjugate axis is a mean proportional between the distances from either of the foci to the extremities of the transverse axis.

That is [Fig. 2],

$$\mathrm{FA} : \mathrm{BC} :: \mathrm{BC} : \mathrm{FA}'.$$

For, $\mathrm{BC}^2 = \mathrm{FB}^2 - \mathrm{CF}^2 = \mathrm{AC}^2 - \mathrm{CF}^2$, [Prop. II.]

or $\mathrm{BC}^2 = (\mathrm{AC}+\mathrm{CF})\times(\mathrm{AC}-\mathrm{CF})$

$= \mathrm{FA}'\times\mathrm{FA}$; and hence,

$$\mathrm{FA} : \mathrm{BC} :: \mathrm{BC} : \mathrm{FA}'.$$

PROPOSITION IV.

THEOREM.

The parameter is less than four times the distance from one of the vertices of the transverse axis to the adjacent focus.

The parameter being denoted by p, then

$$p \text{ is } < 4\,\mathrm{FA}. \qquad \text{[Fig. 2.]}$$

For we have (Def. VII.)

$$2\,\mathrm{AC} : 2\,\mathrm{BC} :: 2\,\mathrm{BC} : p,$$

or $$p = \frac{2\,\mathrm{BC}^2}{\mathrm{AC}}.$$

But (Prop. III.),

$$\mathrm{BC}^2 = \mathrm{FA}\times\mathrm{FA}' = \mathrm{FA}\,(2\,\mathrm{AC}-\mathrm{FA}):$$

hence, by substitution, we get

$$p = \frac{4\,\mathrm{FA}\times\mathrm{AC} - 2\,\mathrm{FA}^2}{\mathrm{AC}}$$

$$= 4\,\mathrm{FA} - \frac{2\,\mathrm{FA}^2}{\mathrm{AC}}.$$

PROPOSITION V.

THEOREM.

The square of an ordinate to the transverse axis, is to the product of the corresponding abscissas, as the square of the conjugate axis, is to the square of the transverse axis.

Let MP [Fig. 3] be an ordinate to the transverse axis; then,

$$\mathrm{MP}^2 : \mathrm{AP}\times\mathrm{A'P} :: \mathrm{BB'}^2 : \mathrm{AA'}^2.$$

Let $\mathrm{AA'} = 2a$, $\mathrm{BB'} = 2b$, $\mathrm{F'C} = \mathrm{FC} = c$; and let $\mathrm{CP} = x$, and $\mathrm{MP} = y$: then,

$$\mathrm{FP} = c-x, \quad \text{and} \quad \mathrm{F'P} = c+x.$$

Also let $\mathrm{F'M} = a+z$, z being a variable quantity; then, $\mathrm{FM} = a-z$.

The right-angled triangles PMF, PMF′ give

$$\mathrm{FM}^2 = \mathrm{MP}^2+\mathrm{FP}^2,$$

and

$$\mathrm{F'M}^2 = \mathrm{MP}^2+\mathrm{F'P}^2;$$

or, substituting,

$$(a-z)^2 = y^2+(c-x)^2,$$

and

$$(a+z)^2 = y^2+(c+x)^2$$

Developing, and subtracting the first equation from the second, we find $z = \frac{cx}{a}$;

and substituting this value of z in the second equation, we have

$$\left(a + \frac{cx}{a}\right)^2 = y^2+(c+x)^2,$$

or

$$a^4+2a^2cx+c^2x^2 = a^2y^2+a^2c^2+2a^2cx+a^2x^2,$$

or

$$a^4-a^2c^2-a^2x^2+c^2x^2 = a^2y^2,$$

or

$$a^2(a^2-c^2)-x^2(a^2-c^2) = a^2y^2,$$

or

$$(a^2-x^2)\times(a^2-c^2) = a^2y^2;$$

and hence, since $a^2 - c^2 = b^2$ (Prop. II.), we have

$$(a^2 - x^2)\,b^2 = a^2y^2. \quad \text{..............} \quad [A]$$

From this we derive the proportion,

$$y^2 : (a^2 - x^2) :: b^2 : a^2;$$

that is, $\quad \text{MP}^2 : \text{AC}^2 - \text{CP}^2 :: \text{BC}^2 : \text{AC}^2, \quad \text{........} \quad [B]$

or $\quad \text{MP}^2 : \text{AC}^2 - \text{CP}^2 :: \text{BB}'^2 : \text{AA}'^2. \quad \text{......} \quad [B']$

But, $\text{AC}^2 - \text{CP}^2 = (\text{AC} + \text{CP}) \times (\text{AC} - \text{CP}) = \text{A}'\text{P} \times \text{AP}$;

hence, $\quad \text{MP}^2 : \text{AP} \times \text{A}'\text{P} :: \text{BB}'^2 : \text{AA}'^2.$

Cor. I. The squares of ordinates to the transverse axis are to each other as the products of the corresponding abscissas.

For, M′P′ being any other ordinate to the transverse axis, we have

$$\text{MP}^2 : \text{AP} \times \text{A}'\text{P} :: \text{BB}'^2 : \text{AA}'^2,$$

and $\quad \text{M}'\text{P}'^2 : \text{AP}' \times \text{A}'\text{P}' :: \text{BB}'^2 : \text{AA}'^2;$

and hence (Geom. II. 4),

$$\text{MP}^2 : \text{M}'\text{P}'^2 :: \text{AP} \times \text{A}'\text{P} : \text{AP}' \times \text{A}'\text{P}'.$$

Cor. II. $\quad \text{MP}^2 : \text{M}'\text{P}'^2 :: \text{AC}^2 - \text{CP}^2 : \text{AC}^2 - \text{CP}'^2.$

Cor. III. $\quad \text{MP}^2 = (\text{AC}^2 - \text{CP}^2)\dfrac{p}{2\,\text{AC}}.$

For, since (Prop. IV.) $p = \dfrac{2\,\text{BC}^2}{\text{AC}}$, we have, by multiplying and dividing the second member by AC,

$$p = \frac{2\,\text{BC}^2 . \text{AC}}{\text{AC}^2};$$

and hence, $\quad p : 2\,\text{AC} :: \text{BC}^2 : \text{AC}^2,$

or (Proportion [B] of this Proposition)

$$p : 2\,\text{AC} :: \text{MP}^2 : \text{AC}^2 - \text{CP}^2,$$

and consequently $\text{MP}^2 = (\text{AC}^2 - \text{CP}^2)\dfrac{p}{2\,\text{AC}}.$

PROPOSITION VI.

THEOREM.

The square of an ordinate to the conjugate axis, is to the product of the corresponding abscissas, as the square of the transverse axis is to the square of the conjugate axis.

Let MP″ [Fig. 3] be an ordinate to the axis BB′; then,

$$MP''^2 : BP'' \times B'P'' :: AA'^2 : BB'^2.$$

Let CP″ $= x'$, and MP″ $= y'$: then

$$x = CP = MP'' = y', \text{ and } y = MP = CP'' = x'.$$

Substituting these values of x and y in Equation [A],

$$a^2y^2 = b^2(a^2-x^2),$$

we have $$a^2x'^2 = b^2(a^2-y'^2);$$

and hence, by transposition,

$$b^2y'^2 = b^2a^2-a^2x'^2 = a^2(b^2-x'^2),$$

which gives $$y'^2 : b^2-x'^2 :: a^2 : b^2,$$

or $$y'^2 : (b+x').(b-x') :: (2a)^2 : (2b)^2;$$

that is, $$MP''^2 : BP'' \times B'P'' :: AA'^2 : BB'^2.$$

Cor. I. The squares of ordinates to the conjugate axis are to each other as the products of the corresponding abscissas.

Cor. II. The ellipse is divided by the conjugate axis into two equal parts, similarly situated with regard to that axis.

PROPOSITION VII.

THEOREM.

The ordinates to the transverse axis of an ellipse, are to the corresponding ordinates of the circle described on that axis as a diameter, as the conjugate axis is to the transverse axis.

Let M′P [Fig. 4] be an ordinate of the circumference described on AA′ as a diameter; then,

$$MP : M'P :: BB' : AA'.$$

For, by Prop. V. we have

$$MP^2 : AP \times A'P :: BB'^2 : AA'^2;$$

but (Geom. IV. 23, cor.),

$$AP : M'P :: M'P : A'P;$$

hence $$AP \times A'P = M'P^2,$$

and consequently

$$MP^2 : M'P^2 :: BB'^2 : AA'^2,$$

or $$MP : M'P :: BB' : AA'.$$

Cor. I. Similar (but inverse) relations may be shown to exist between the ordinates to the conjugate axis, and those of the circumference described on that axis.

Cor. II. When the axes of an ellipse are given, any number of points of the curve may be determined by the following construction :

On AA′ and BB′ the transverse and conjugate axes as diameters, describe the circumferences of two circles. Let M′P be an ordinate of the larger circumference, and draw M′C cutting the smaller in G; then draw GM parallel to AA′. M is a point of the ellipse.

For, the similar triangles GMM′ and CPM′ give

$$MP : M'P :: CG : CM',$$

or

$$MP : M'P :: CB : CA,$$

or

$$MP : M'P :: BB' : AA'.$$

Any number of points may thus be determined.

PROPOSITION VIII.

THEOREM.

The parameter is equal to the double ordinate to the transverse axis, drawn through one of the foci.

That is, NN′ [Fig. 3] being the double ordinate to AA′ drawn through the focus F,

$$AA' : BB' :: BB' : NN'.$$

We have (Proposition V. Proportion [B′]),

$$MP^2 : AC^2-CP^2 :: BB'^2 : AA'^2;$$

consequently, when the abscissa CP becomes CF,

$$FN^2 : CA^2-CF^2 :: BB'^2 : AA'^2;$$

but (Prop. II.), $CA^2-CF^2 = BF^2-CF^2 = BC^2$;

hence

$$FN^2 : BC^2 :: BB'^2 : AA'^2,$$

and

$$FN : BC :: BB' : AA';$$

or, inverting, and doubling the first and second terms,

$$AA' : BB' :: BB' : NN';$$

and therefore (Def. VII.) $NN' = p$, the parameter.

PROPOSITION IX.

THEOREM.

The sum of the lines drawn from the foci to a point in the plane of the curve, is greater than the transverse axis if the point be without the curve, but less if it be within.

1. Let R′ [Fig. 5] be any point without the ellipse; then, since FR′+MR′ is > FM,

$$FR'+F'R' \text{ is } > FM+F'M > AA'.$$

2. Let R be any point within the ellipse; then, since FR is < FM+MR,

$$FR+F'R \text{ is } < FM+F'M < AA'.$$

COR. A point is without or within the ellipse, according as the sum of the lines drawn from it to the foci is greater or less than the transverse axis.

PROPOSITION X.

THEOREM.

The straight line which bisects the angle adjacent to the angle contained by the radii vectores of any point of the ellipse, is a tangent to the curve at that point.

M [Fig. 6] being any point in the curve, if the line DMN be drawn so as to bisect the angle FMI adjacent to the angle FMF′, then will every point of that line, except M, be without the curve; and consequently DMN will be a tangent to the curve at M.

Let D be any point of the line DN, except M : make MI=MF, and draw the straight lines FI, DI, DF and DF′.

Then, DN, which bisects the angle FMI of the isosceles triangle FMI, also bisects the base FI at right angles: hence the triangle FDI is isosceles, and we have

$$DF = DI; \text{ and therefore}$$

$$F'D+FD = F'D+DI;$$

but, $F'D+DI$ is $> F'I > F'M+FM > AA'$;

hence $F'D+FD$ is $> AA'$, and (Prop. IX. cor.) the point D is without the ellipse.

Cor. I. From the above, the method of drawing a tangent to the ellipse, at any point M of the curve, is obvious.

Cor. II. If it be required to draw a tangent to the ellipse, which shall pass through a given point D without the curve; then from F′ and D [Fig. 7] as centres, and with radii equal respectively to AA′ and FD, describe two arcs of circles; and from either of their points of intersection, as I, draw F′I meeting the curve in M : NMD is the tangent required.

For, since $F'I = AA'$, and $F'M+FM = AA'$,

we have $F'I = F'M+FM$,

or $F'M+MI = F'M+FM$;

and hence $MI = FM$,

and by construction we have

$$DF = DI.$$

Hence it may easily be shown that DN bisects the angle FMI : it is therefore a tangent at M.

In like manner it may be shown that DM′ is a tangent to the curve at M′. Hence two tangents may be drawn to the ellipse, from a given point without it.

Cor. III. The angles made by a tangent to the ellipse, with the radii vectores of the point of contact, are equal.

For $\qquad FMN = NMI,$

and the opposite vertical angles NMI and F′MD are equal;

hence, $\qquad FMN = F'MD.$

Cor. IV. The tangents at the extremities of the transverse and conjugate axes are respectively perpendicular to these axes.

PROPOSITION XI.

THEOREM.

If, from the foci, perpendiculars be drawn to a tangent at any point of the ellipse, they will meet it in the circumference of the circle described on the transverse axis as a diameter.

In addition to the preceding construction, draw F′N′ perpendicular to NN′; then are the points N and N′ in the circumference described on AA′ as a diameter.

Join the points C and N, and produce CN and F′N′ till they meet in D.

Since $FC = F'C$ and $FN = NI$, CN is parallel to F′I (Geom. IV. 16), and the triangles F′FI and CFN are similar; therefore $CN = \frac{1}{2} F'I = \frac{1}{2} AA' = AC$.

Again, since DN and F′I are parallel to each other, and also FN and F′N′, they being perpendicular to the same straight line NN′, the figure DNIF′ is a parallelogram, and

$$DN = F'I = 2\,AC;$$

but CN = AC, therefore DC = AC.

Hence, if from C as a centre, and with CA as a radius, the circumference of a circle be described, it will pass through the points N and D; and since NN′D is a right angle, it will also pass through N′ (Geom. III. 18, cor. 2).

PROPOSITION XII.

THEOREM.

If, from the foci, perpendiculars be drawn to a tangent at any point of the curve, then will the product of the perpendiculars be equal to the square of the semi-conjugate axis.

Let the construction be the same as in the preceding proposition; then [Fig. 8],

$$FN \times F'N' = BC^2.$$

For (Geom. IV. 28, cor.)

$$F'N' \times F'D = A'F' \times F'A;$$

but, since DNIF′ is a parallelogram,

$$F'D = IN = FN,$$

and (Prop. III.) $A'F' \times F'A = BC^2;$

hence, by substitution,

$$F'N' \times FN = BC^2.$$

Cor. In addition to the preceding construction, let FN″ be drawn perpendicularly to the tangent at M′; then,

$$FN : FN'' :: \sqrt{\frac{FM}{F'M}} : \sqrt{\frac{FM'}{F'M'}}.$$

For, the right-angled triangles FMN and F′MN′ having the angles FMN and F′MN′ equal (Prop. X. cor. 3), are similar, and give

$$\mathrm{FN} : \mathrm{FM} :: \mathrm{F'N'} : \mathrm{F'M};$$

hence,
$$\mathrm{FN} = \frac{\mathrm{FM}\times\mathrm{F'N'}}{\mathrm{F'M}};$$

or, multiplying each member of this equation by FN,

$$\mathrm{FN}^2 = \mathrm{FN}\times\mathrm{F'N'}\times\frac{\mathrm{FM}}{\mathrm{F'M}}$$

$$= \mathrm{BC}^2\times\frac{\mathrm{FM}}{\mathrm{F'M}},$$

and
$$\mathrm{FN} = \mathrm{BC}.\sqrt{\frac{\mathrm{FM}}{\mathrm{F'M}}};$$

hence, for the perpendicular FN″ upon the tangent at M′, we shall have

$$\mathrm{FN''} = \mathrm{BC}.\sqrt{\frac{\mathrm{FM'}}{\mathrm{F'M'}}},$$

and consequently

$$\mathrm{FN} : \mathrm{FN''} :: \sqrt{\frac{\mathrm{FM}}{\mathrm{F'M}}} : \sqrt{\frac{\mathrm{FM'}}{\mathrm{F'M'}}}.$$

PROPOSITION XIII.

PROBLEM.

To find the algebraic expression of the radii vectores of the ellipse.

From the foci, draw the lines FM and F′M [Fig. 9] to any point M of the ellipse : Required the algebraic expressions of these lines.

As in the fifth proposition,
let AC $= a$, BC $= b$, FC $= c$, and CP $= x$;
also let F′M $= a+z$, giving FM $= a-z$:

then, since $z = \frac{cx}{a}$, (Prop. V.)

we have, by substitution,

$$\text{F}'\text{M} = a+\frac{cx}{a} = \frac{a^2+cx}{a},$$

and

$$\text{FM} = a-\frac{cx}{a} = \frac{a^2-cx}{a}.$$

Cor. The right-angled triangle F′MP gives

$$\text{F}'\text{P} = \text{F}'\text{M} \times \cos \text{MF}'\text{P} :$$ we also have

$$\text{F}'\text{P} = \text{F}'\text{C}+\text{CP} = c+x;$$

hence $\text{F}'\text{M} \times \cos \text{MF}'\text{P} = c+x,$

and $x = \text{F}'\text{M} \times \cos \text{MF}'\text{P}-c.$

Hence $\text{F}'\text{M} = a+\frac{cx}{a} = a+\frac{c}{a}(\text{F}'\text{M}.\cos \text{MF}'\text{P} -c),$

and $\text{F}'\text{M}\times a = a^2+c.\text{F}'\text{M}.\cos \text{MF}'\text{P} -c^2,$

or $\text{F}'\text{M}(a-c.\cos \text{MF}'\text{P}) = a^2-c^2,$

and $\text{F}'\text{M} = \frac{a^2-c^2}{a-c.\cos \text{MF}'\text{P}}.$

This is called the *polar equation* of the ellipse.

PROPOSITION XIV.

PROBLEM.

To find the algebraic expression of the subnormal of the ellipse.

Draw MQ [Fig. 9] perpendicular to the tangent TT′ at the point of contact M; then MQ is the normal, and

PQ is the subnormal to the ellipse at the point M : Required the algebraic expression of the latter line.

The straight lines FN and QM, being perpendicular to the tangent TT′, are therefore parallel to each other, and give (Geom. IV. 15, cor. 1)

$$\text{F}'\text{I} : \text{F}'\text{F} :: \text{MI}=\text{MF} : \text{QF},$$

or
$$2a : 2c :: \frac{a^2-cx}{a} : \text{QF};$$

hence,
$$\text{QF} = 2c \,.\, \frac{a^2-cx}{2a^2} = \frac{a^2c-c^2x}{a^2}.$$

We also have $\text{FP} = c-x;$
but the subnormal $\text{PQ} = \text{FQ}-\text{FP};$
and hence, by substitution, we get

$$\text{PQ} = \frac{a^2c-c^2x}{a^2} - c + x$$

$$= \frac{a^2c-c^2x-a^2c+a^2x}{a^2}$$

$$= \frac{a^2x-c^2x}{a^2} = \frac{(a^2-c^2)x}{a^2} = \frac{b^2}{a^2}x.$$

Cor. Denoting the parameter by p, we have

$$2a : 2b :: 2b : p,$$

and
$$p = \frac{4b^2}{2a};$$

and dividing each member of this equation by $2a$, we find

$$\frac{p}{2a} = \frac{b^2}{a^2}.$$

Hence
$$\text{PQ} = \frac{px}{2a},$$

and
$$2a : p :: x : \text{PQ},$$

or
$$\text{AA}' : \text{N}'\text{N}'' :: \text{CP} : \text{PQ}.$$

Therefore, as CP is known when M is given, in order to determine the subnormal for a given point M, we have only to find a fourth proportional (Geom. IV·2) to the lines AA′, N′N″ and CP.

Remark. The method of drawing a normal or tangent to the curve, by means of the subnormal, is obvious.

PROPOSITION XV.

PROBLEM.

To find the algebraic expression of the subtangent of the ellipse.

MT [Fig. 9] being a tangent at the point M, and MP the corresponding ordinate to the transverse axis : Required the algebraic expression of the line PT.

Since MP is perpendicular to QT the hypothenuse of the right-angled triangle QMT, we have (Geom. IV. 23)

$$\mathrm{PQ} : \mathrm{MP} :: \mathrm{MP} : \mathrm{PT},$$

or

$$\frac{b^2}{a^2}x : y :: y : \mathrm{PT};$$

and hence

$$\mathrm{PT} = \frac{y^2}{\frac{b^2}{a^2}x};$$

but,

$$y^2 = \frac{b^2}{a^2}(a^2-x^2); \qquad \text{(Prop. V.)}$$

hence,

$$\mathrm{PT} = \frac{\frac{b^2}{a^2}(a^2-x^2)}{\frac{b^2}{a^2}x} = \frac{a^2-x^2}{x}.$$

Cor. I. $\mathrm{PT} = \frac{a^2-x^2}{x} = \frac{(a+x).(a-x)}{x}$.

and hence $x : a+x :: a-x : \mathrm{PT}$,

that is, $\mathrm{CP} : \mathrm{A'P} :: \mathrm{AP} : \mathrm{PT}$.

The point M being given, and consequently CP, A′P and AP being known, the subtangent PT may thus be determined by finding a fourth proportional to these lines.

Remark. The method of drawing a tangent to the ellipse, by means of the subtangent, is obvious.

Cor. II. If on AA′ [Fig. 10] as a transverse axis, different ellipses ABA′, AB′A′, AB″A′, &c. be described, and at the points M, M′, M″, &c. which have a common abscissa CP $= x$, tangents be drawn, they will intersect AA′ produced in the same point T.

For as the expression of the subtangent $\frac{a^2-x^2}{x}$ involves only a and x, it remains the same while a and x remain the same, however b may vary.

Cor. III. If a tangent to the ellipse meet the transverse axis produced, and from the point of contact an ordinate be drawn to the same axis; then is the semi-transverse axis a mean proportional between the distances from the centre to the intersections of the axis by the ordinate and tangent. That is [Fig. 9],

$$\mathrm{CP} : \mathrm{CA} :: \mathrm{CA} : \mathrm{CT}.$$

For, denoting the subtangent PT by s, we have

$$sx = a^2-x^2,$$

or $a^2 = x^2+sx = x(x+s)$;

and hence $x : a :: a : x+s$,

that is, $\mathrm{CP} : \mathrm{AC} :: \mathrm{AC} : \mathrm{CT}$.

COR. IV. The equation $PT = \frac{a^2-x^2}{x}$ gives

$$PT = \frac{AC^2-CP^2}{CP}.$$

PROPOSITION XVI.

PROBLEM.

To find the algebraic expression of the subtangent, measured on the conjugate axis.

MT [Fig. 9] being produced to meet the axis BB′ in T′, and the ordinate MP′ being drawn : Required the algebraic expression of the line P′T′.

The similar triangles MPT and MP′T′ give

$$PT : MP :: MP'=CP : P'T',$$

or $$\frac{a^2-x^2}{x} : y :: x : P'T':$$

hence $$P'T' = \frac{yx}{\frac{a^2-x^2}{x}} = \frac{yx^2}{a^2-x^2};$$

but the equation $a^2y^2 = b^2(a^2-x^2)$ (Prop. V. [A]) gives

$$a^2-x^2 = \frac{a^2y^2}{b^2}, \text{ and } x^2 = \frac{a^2b^2-a^2y^2}{b^2}:$$

substituting these values of x^2 and a^2-x^2 in the expression for P′T′, we have

$$P'T' = \frac{y.\frac{a^2b^2-a^2y^2}{b^2}}{\frac{a^2y^2}{b^2}} = \frac{y(a^2b^2-a^2y^2)}{a^2y^2}$$

$$= \frac{b^2-y^2}{y}.$$

Cor. I. $CP' : B'P' :: BP' : P'T'$.

Cor. II. The tangents at the corresponding points of all the ellipses described on BB' as a conjugate axis, meet that axis produced in the same point.

Cor. III. $CP' : CB :: CB : CT'$.

For, denoting $P'T'$ by s', we have

$$s' = \frac{b^2-y^2}{y}, \quad \text{and } b^2 = y^2+s'y$$
$$= y(y+s');$$

and hence $y : b :: b : y+s'$,

that is, $CP' : CB :: CB : CT'$.

OF DIAMETERS.

Def. IX. Every line drawn through the centre, and terminated by the curve, is called a *diameter*.

Thus [Fig. 1], MM' is a diameter of the ellipse $ABA'B'$.

A second diameter drawn parallel to the tangent at one of the extremities of the first, is called the *conjugate diameter* to the first.

Thus, NN', parallel to the tangent at M, is the conjugate diameter to MM'.

X. An *ordinate* to a diameter, is a straight line drawn from any point in the curve, parallel to the tangent at one of the extremities of that diameter, and terminated by it. The parts into which the diameter is divided by the ordinate, are called *abscissas*.

Thus, $M''O$, parallel to the tangent at M, is an ordinate to MM', and MO and $M'O$ are the corresponding abscissas.

PROPOSITION XVII.

THEOREM.

If a radius vector be drawn from the extremity of any diameter, to meet its conjugate, the part intercepted by the conjugate is equal to the semi-transverse axis.

If the radius vector MF [Fig. 11] be drawn from M the extremity of any diameter MM′, the part MG intercepted by the conjugate NN′ is equal to AC.

For (drawing F′D parallel to NN′), since the equal angles F′MT, FMT′ (Prop. X. cor. 3) are equal respectively to the alternate interior angles MF′D, MDF′, the triangle MF′D is isosceles, and MD = MF′. Also since GC is parallel to the side F′D of the triangle F′DF, and F′C = FC, we have

$$DG = GF.$$

Consequently $F'M+FM = 2\,MD+2\,DG = 2\,MG.$

But, $F'M+FM = 2\,AC;$

hence $2\,MG = 2\,AC,$

or $MG = AC.$

PROPOSITION XVIII.

THEOREM.

Every diameter is bisected at the centre.

From M [Fig. 12] any point of the curve, draw the radii vectores MF, MF′; also draw M′F, M′F′ parallel respectively to MF′ and MF : then is the figure MFM′F′ a parallelogram, and $M'F+M'F' = MF'+MF = AA'$;
hence M′ is a point in the ellipse. But the diagonals of a parallelogram bisect each other; therefore FF′ is bisected at C, and C is the centre of the ellipse, and MM′, which is also bisected at C, is a diameter.

PROPOSITION XIX.

THEOREM.

If a tangent be drawn at the extremity of any diameter, and also at the adjacent extremity of the transverse axis, and from the first point an ordinate be drawn to that axis; then, the quadrilateral formed by the intersections of the axis and the diameter produced, by the ordinate and the tangent at the second point, is equivalent to the triangle formed by the mutual intersection of the axis produced, the ordinate, and the tangent at the first point.

MM‴ [Fig. 13] being any diameter, let the tangents MT and AR, and the ordinate MP, be drawn; then,

$$\text{PMRA} = \text{PMT}.$$

The similar triangles CPM and CAR give

$$\text{PC} : \text{AC} :: \text{PM} : \text{AR};$$

but (Prop. XV. cor. 3),

$$\text{PC} : \text{AC} :: \text{AC} : \text{CT};$$

therefore $$\text{PM} : \text{AR} :: \text{AC} : \text{CT},$$

and $$\frac{\text{AC}\times\text{AR}}{2} = \frac{\text{CT}\times\text{PM}}{2};$$

that is, the triangle CAR = the triangle CTM,

or $\text{CPM}+\text{PMRA} = \text{CPM}+\text{PMT}$,

and consequently $\text{PMRA} = \text{PMT}$.

PROPOSITION XX.

THEOREM.

If, from the extremities of a double ordinate to any diameter, ordinates be drawn to the transverse axis, and also a tangent at the adjacent extremity of the diameter; then, the triangles formed by the intersections of the ordinates and the diameter produced if necessary, are equal to each other; and each is also equivalent to the quadrilateral resulting from the intersections of the diameter and the axis produced, by the tangent and the ordinate to the diameter also produced.

From M′, M″ [Fig. 13] the extremities of a double ordinate to any diameter MM‴, let the ordinates M′P′ and M″P″ to the transverse axis be drawn, and also TT′ the tangent to the ellipse at M; then, the triangles M′OK and M″OK′ are equal to each other, and each is equivalent to the quadrilateral MOST.

FIRST PART.

The similar triangles PMT, P′M′S give (Geom. IV. 25),

$$PMT : P'M'S :: MP^2 : M'P'^2;$$

but (Prop. V. cor. 2),

$$MP^2 : M'P'^2 :: AC^2 - CP^2 : AC^2 - CP'^2;$$

hence $PMT : P'M'S :: AC^2 - CP^2 : AC^2 - CP'^2$. .... [C]

Again, the similar triangles CAR, CPM give

$$CAR : CPM :: CA^2 : CP^2;$$

and hence (Geom. II. 6)

$$CAR - CPM : CAR :: CA^2 - CP^2 : CA^2,$$

or $PARM : CAR :: CA^2 - CP^2 : CA^2$. .... [D]

Also the similar triangles CAR, CP'K give

$CAR : CP'K :: CA^2 : CP'^2$; and

hence $CAR - CP'K : CAR :: CA^2 - CP'^2 : CA^2$,

or $P'ARK : CAR :: CA^2 - CP'^2 : CA^2$. ....[E]

And comparing Proportions [D] and [E], we have

$PARM : P'ARK :: CA^2 - CP^2 : CA^2 - CP'^2$;

but (Proportion [C]),

$PMT : P'M'S :: CA^2 - CP^2 : CA^2 - CP'^2$;

hence, $PMT : P'M'S :: PARM : P'ARK$;

and since (Prop. XIX.) $PMT = PARM$, we have

$P'M'S = P'ARK$. .... [F]

But, $P'ARK = P'PMK + PARM$

$= P'PMK + PMT$

$= P'TMK$;

consequently (Equation [F]),

$P'M'S = P'ARK = P'TMK$.

But $P'M'S = M'OK + P'KOS$,

and $P'TMK = MOST + P'KOS$;

therefore, $M'OK = MOST$.

SECOND PART.

The similar triangles PMT, P''M''S give

$PMT : P''M''S :: MP^2 : M''P''^2$;

but, $MP^2 : M''P''^2 :: CA^2 - CP^2 : CA^2 - CP''^2$;

hence $PMT : P''M''S :: CA^2 - CP^2 : CA^2 - CP''^2$..[C']

Again, the similar triangles CAR, CPM give, as in the preceding part of the demonstration,

$PARM : CAR :: CA^2 - CP^2 : CA^2$. ...[D']

Also, the similar triangles CAR, CP''K' give

$CAR : CP''K' :: CA^2 : CP''^2$; and

hence $P''ARK' : CAR :: CA^2-CP''^2 : CA^2$. ..[E′]

Comparing Proportions [D′] and [E′], we have

$$PARM : P''ARK' :: CA^2-CP^2 : CA^2-CP''^2;$$

and comparing this proportion with Proportion [C′], we have $PMT : P''M''S :: PARM : P''ARK'$;

but $PTM = PARM$; (Prop. XIX.)

hence, $P''M''S = P''ARK'$. .... [F′]

Now we have (Equation [F]),

$$P'M'S = P'ARK;$$

and subtracting Equation [F′] from this, we get

$$P'M'S-P''M''S = P'ARK-P''ARK',$$

or $P'M'M''P'' = P'P''K'K$;

but $P'M'M''P'' = P'KOM''P''+M'OK$,

and $P'P''K'K = P'KOM''P''+M''OK'$;

hence $M'OK = M''OK'$,

and therefore $M''OK' = MOST$.

Cor. I. Every diameter bisects its double ordinates.

For, the triangles M′OK, M″OK′ being equivalent and similar, all their parts are equal, and hence

$$M'O = M''O.$$

Cor. II. If, from N the extremity of the conjugate to MM‴, ND be drawn parallel to BB′, then will the triangles CTM, CND be equivalent to each other.

Let M′M″ move parallel to itself till it coincides with NN′; then the points M′, O and K coincide respectively with the points N′, C and D′, and the triangle M′OK becomes the triangle N′CD′; also the points O and S coincide with C, and MOST becomes MCT, and hence $N'CD' = MCT$. But $N'CD' = NCD$, since they are similar, and $NC = N'C$; and therefore $NCD = MCT$.

PROPOSITION XXI.

THEOREM.

The square of an ordinate to any diameter, is to the product of the corresponding abscissas, as the square of the conjugate to that diameter, is to the square of the diameter itself.

That is [Fig. 13],

$$M'O^2 : M'''O \times MO :: NN'^2 : MM'''^2.$$

The similar triangles M′OK, NCD give

$$M'OK : NCD :: M'O^2 : NC^2. \quad \ldots\ldots \quad [G]$$

Also the similar triangles CMT, COS give

$$CM^2 : CO^2 :: CMT : COS,$$

or $$CM^2 - CO^2 : CM^2 :: MOST : CMT;$$

or since MOST = M′OK, and CMT = CND,

$$CM^2 - CO^2 : CM^2 :: M'OK : CND;$$

and comparing this proportion with Proportion [G], we have

$$CM^2 - CO^2 : CM^2 :: M'O^2 : CN^2,$$

or $$(CM + CO) \times (CM - CO) : CM^2 :: M'O^2 : CN^2,$$

or $$M'O^2 : M'''O \times MO :: CN^2 : CM^2,$$

or $$M'O^2 : M'''O \times MO :: NN'^2 : MM'''^2.$$

Cor. I. Let M″O′ [Fig. 14] be any other ordinate to the diameter MM‴; then,

$$M'O^2 : M''O'^2 :: M'''O \times MO : M'''O' \times MO'.$$

Cor. II. If CO = CO′, then M′O = M″O′.

Cor. III. Tangents to the ellipse, at the extremities of a diameter, are parallel to each other.

Let any line SS′ parallel to the tangent at M, move parallel to itself till it pass through the point M‴; then, since (Prop. XX. cor. 1) the intercepted parts M′O, M^{iv}O remain constantly equal, they will vanish at the same time at M‴, and the line in its new position TT′ will evidently be a tangent to the ellipse.

Cor. IV. A tangent to the ellipse at the extremity N of the conjugate diameter NN′, is parallel to the first diameter MM‴.

Let CO, CO′ be any two equal abscissas, and draw the ordinates M′O, M″O′ to MM‴; then M′O, M″O′ being (Cor. II.) equal as well as parallel, the quadrilateral M′OO′M″ is a parallelogram, and M′Q = M″Q. Then, reasoning as in the preceding corollary, it may be shown that the tangent at N′ is parallel to MM‴.

Cor. V. Since MM‴ is parallel to the tangent at N′, it is (Def. IX.) a conjugate to NN′: hence MM‴ and NN′ are conjugate to each other.

PROPOSITION XXII.

THEOREM.

If, from the extremities of a pair of semi-conjugate diameters, two ordinates be drawn to the transverse axis; then the sum of the squares of the parts of the axis comprehended between the centre and the ordinates, is equal to the square of half that axis.

That is, if CN and CM [Fig. 15] be semi-conjugate diameters, and the ordinates MP, NZ be drawn to AA′, then $CP^2+CZ^2 = AC^2$.

The similar triangles MPT, N′P′C give

$$TP^2 : PM^2 :: CP'^2 : P'N'^2;$$

but (Prop. XV. cor. 4), $TP^2 = \frac{(AC^2-CP^2)^2}{CP^2}$,

and (Prop. V. cor. 3), $PM^2 = (AC^2-CP^2)\frac{p}{2\,AC}$,

and $P'N'^2 = (AC^2-CP'^2)\frac{p}{2\,AC}$;

hence, by substitution,

$$\frac{(AC^2-CP^2)^2}{CP^2} : (AC^2-CP^2)\frac{p}{2\,AC} :: CP'^2 : (AC^2-CP'^2)\frac{p}{2\,AC},$$

or $\frac{(AC^2-CP^2)^2}{CP^2} : AC^2-CP^2 :: CP'^2 : AC^2-CP'^2$,

or $AC^2-CP^2 : CP^2 :: CP'^2 : AC^2-CP'^2$,

or $AC^2 : CP^2 :: AC^2 : AC^2-CP'^2$,

and hence $CP^2 = AC^2-CP'^2$;

and since $CP' = CZ$, we have

$$CP^2 = AC^2-CZ^2,$$

or $CP^2+CZ^2 = AC^2$.

PROPOSITION XXIII.

THEOREM.

The parallelogram formed by drawing tangents to the ellipse at the extremities of a pair of conjugate diameters, is equal to the rectangle of the axes.

That is, the parallelogram EE′E″E‴ [Fig. 16] is equal to the rectangle HH′H″H‴.

NZ being an ordinate to the transverse axis, we have (Prop. V. [B]),

$$NZ^2 : AC^2 - CZ^2 :: BC^2 : AC^2;$$

or, since (Prop. XXII.) $AC^2 - CZ^2 = CP^2$,

$$NZ^2 : CP^2 :: BC^2 : AC^2;$$

and hence $$NZ \times AC = CP \times BC.$$

Now let the perpendicular CY′ be drawn to the tangent TT′ at M : The triangles TCY′, CNZ are similar, because each has a right angle, and the angles NCZ and CTY′ are equal, they being alternate interior angles, and therefore give

$$CN : NZ :: TC : CY',$$

and hence $$CN \times CY' = TC \times NZ.$$

But (Prop. XV. cor. 3), $TC = \frac{AC^2}{CP}$;

hence $$CN \times CY' = \frac{AC^2 \times NZ}{CP}$$

$$= \frac{AC \times AC \times NZ}{CP}:$$

but we have found $AC \times NZ = CP \times BC$;

hence $$\mathrm{CN}\times\mathrm{CY}' = \frac{\mathrm{AC}\times\mathrm{CP}\times\mathrm{BC}}{\mathrm{CP}}$$
$$= \mathrm{AC}\times\mathrm{BC},$$

and $4\,\mathrm{CN}\times\mathrm{CY}' = 2\,\mathrm{AC}\times2\,\mathrm{BC} = \mathrm{AA}'\times\mathrm{BB}'$.
Now $\mathrm{CN}\times\mathrm{CY}'$ = the parallelogram CNEM, and
hence $4\,\mathrm{CN}\times\mathrm{CY}'$ = the parallelogram EE′E″E‴ :
also $\mathrm{AA}'\times\mathrm{BB}'$ = the rectangle HH′H″H‴;
consequently,
parallelogram EE′E″E‴ = rectangle HH′H″H‴.

Cor. I. If MO be drawn perpendicular to NN′, then, since MO = CY′, we have
$$\mathrm{MO}\times\mathrm{CN} = \mathrm{BC}\times\mathrm{AC},$$
and $$\mathrm{CN}^2 = \frac{\mathrm{BC}^2\times\mathrm{AC}^2}{\mathrm{MO}^2}.$$

Cor. II. Let FZ′ and F′Y be drawn perpendicular to TT′; then, from the similar triangles FMZ′, F′MY, MGO, we have
$$\mathrm{F'M} : \mathrm{F'Y} :: \mathrm{MG} : \mathrm{MO},$$
and $$\mathrm{FM} : \mathrm{FZ'} :: \mathrm{MG} : \mathrm{MO};$$
and hence (Geom. II. 13),
$$\mathrm{FM}\times\mathrm{F'M} : \mathrm{FZ'}\times\mathrm{F'Y} :: \mathrm{MG}^2 : \mathrm{MO}^2 :$$
but (Prop. XII.) $\mathrm{FZ'}\times\mathrm{F'Y} = \mathrm{BC}^2$,

and (Prop. XVII.) $\mathrm{MG} = \mathrm{AC}$;

hence $$\mathrm{FM}\times\mathrm{F'M} : \mathrm{BC}^2 :: \mathrm{AC}^2 : \mathrm{MO}^2,$$
and therefore $$\mathrm{FM}\times\mathrm{F'M} = \frac{\mathrm{BC}^2\times\mathrm{AC}^2}{\mathrm{MO}^2}$$
$$= \mathrm{CN}^2. \quad \text{(Cor. I.)}$$

PROPOSITION XXIV.

THEOREM.

The sum of the squares of a pair of conjugate diameters, is equal to the sum of the squares of the axes.

That is [Fig. 16],

$$MM'''^2+NN'^2 = AA'^2+BB'^2.$$

The triangle MCF′ gives (Geom. IV. 13)

$$MF'^2 = MC^2+CF'^2+2\,CP\times CF',$$

and the triangle MCF gives (Geom. IV. 12)

$$MF^2 = MC^2+CF^2-2\,CP\times CF.$$

Adding these equations, and observing that $CF = CF'$, we have $\quad MF^2+MF'^2 = 2\,MC^2+2\,CF^2$

and hence $\quad 2\,MC^2 = MF^2+MF'^2-2\,CF^2.$

Now (Prop. XXIII. cor. 2), $2\,CN^2 = 2\,MF\times MF'$;

hence, by addition, we have

$$\begin{aligned} 2\,MC^2+2\,NC^2 &= (MF+MF')^2-2\,CF^2 \\ &= 4\,AC^2-2\,CF^2 \quad \text{(Prop. I. cor. 1.)} \\ &= 4\,AC^2-2\,(BF^2-BC^2) \\ &= 4\,AC^2-2\,AC^2+2\,BC^2 \\ &= 2\,AC^2+2\,BC^2, \quad \text{and consequently} \end{aligned}$$

$$MM'''^2+NN'^2 = AA'^2+BB'^2.$$

CHAPTER II.

THE PARABOLA.

DEFINITIONS.

I. The Parabola is a curve in which every point is equally distant from a fixed point called the *focus*, and a straight line also fixed, called the *directrix*.

Thus [Fig. 17], if F be the focus and EG the directrix of a parabola MAM″; and if from any point M of the curve, MF and MD be drawn, the one to the focus, the other perpendicular to the directrix, then

$$MF = MD.$$

II. The straight line drawn through the focus, perpendicular to the directrix, is called the *axis*. The point in which the axis intersects the curve, is called the *vertex* of the parabola.

Thus, BX, drawn through F, perpendicular to EG, is the axis; and A, the middle (Def. I.) of the perpendicular FB, is the vertex of the parabola MAM″.

III. A straight line drawn from any point in the curve, perpendicular to the axis, is called an *ordinate* to the axis. The part of the axis intercepted by the vertex and ordinate, is called an *abscissa*.

Thus, MP, perpendicular to BX, is an ordinate to BX; and AP is the corresponding abscissa.

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IV. A straight line equal to four times the distance from the vertex of the curve to the directrix, is called the *parameter* of the axis. It is denoted by p.

It will be shown hereafter (Prop. II. Cor. I.) that the parameter is a third proportional to any abscissa and the corresponding ordinate.

PROPOSITION I.

PROBLEM.

To describe a parabola.

FIRST METHOD : BY POINTS.

Let F [Fig. 17] be the focus, and EG the directrix of a parabola. Through F draw BX perpendicular to EG, and bisect BF; A, the point of bisection, is the vertex of the curve. To find other points, draw a perpendicular MM″ to the axis, cutting it in any point beyond A; and from F as a centre, and with PB as a radius, describe an arc of a circle : the points M and M″ in which this arc cuts the perpendicular, are points of the curve.

For, $$FM = FM'' = PB = MD$$

Any number of points of the parabola may thus be determined.

Cor. Every straight line, as MM″, terminated by the curve, and perpendicular to the axis, is bisected by it; and consequently the parabola consists of two equal branches, similarly placed with regard to the axis.

SECOND METHOD : BY CONTINUOUS MOTION.

Let GDX′ be a rule consisting of two branches DG and DX′ at right angles with each other, and let the extremities of a thread equal in length to the branch DX′ be fixed at the points F and X′; then let the rule be moved in such a manner that the branch DG shall coincide with the directrix, whilst the thread is kept constantly tense by applying a portion of it to DX′ by means of a pencil : the point M of the pencil will describe a parabola. For in every position of the rule,

$$\mathrm{FM}+\mathrm{MX}' = \mathrm{DM}+\mathrm{MX}';$$

and hence $\mathrm{FM} = \mathrm{DM}$.

PROPOSITION II.

THEOREM.

The square of an ordinate to the axis, is equal to the rectangle of the parameter and the corresponding abscissa.

Let MP be any ordinate to the axis; then,

$$\mathrm{MP}^2 = p.\mathrm{AP}.$$

Let [Fig. 17] $\mathrm{AP} = x$, $\mathrm{MP} = y$, $\mathrm{AB} = \mathrm{AF} = c$;
then (Def. IV.) $p = 4\,\mathrm{AB} = 4\,c$,

$$\mathrm{FM} = \mathrm{MD} = \mathrm{AP}+\mathrm{AB} = x+c,$$

and $\mathrm{FP} = \mathrm{AP}-\mathrm{AF} = x-c$.

Since FPM is right-angled at P,

$$\mathrm{MP}^2 = \mathrm{FM}^2-\mathrm{FP}^2,$$

or

$$\begin{aligned} y^2 &= (x+c)^2-(x-c)^2 \\ &= x^2+2cx+c^2-x^2+2cx-c^2 \\ &= 4\,cx, \quad \ldots\ldots\ldots\ldots\ldots \quad [\mathrm{A}] \end{aligned}$$

or $y^2 = px$;

that is, $\mathrm{MP}^2 = p.\mathrm{AP}$.

COR. I. $AP : MP :: MP : p.$

COR. II. The squares of ordinates to the axis, are to each other as the corresponding abscissas.

For, MP and M′P′ being any two ordinates to the axis, $MP^2 = p.AP$, and $M'P'^2 = p.AP'$; and hence,

$$MP^2 : M'P'^2 :: AP : AP'.$$

COR. III. The parameter is equal to twice the focal ordinate to the axis.

For, Equa. [A], $y^2 = 4cx$, gives

$$MP^2 = 4\,AF \times AP;$$

and when AP becomes AF, MP becomes $M^{iv}F$, and hence

$$M^{iv}F^2 = 4\,AF^2,$$

and $$M^{iv}F = 2\,AF,$$

and $$M^{iv}M^{v} = 4\,AF = p. \qquad \text{(Def. IV.)}$$

COR. IV. The two branches of the parabola constantly diverge from the axis; for as AP increases, MP also increases. From this it is evident that a line drawn perpendicular to the directrix, and produced sufficiently far, will meet the curve.

PROPOSITION III.

THEOREM.

The distance from any point in the plane of the parabola to the focus, is greater than its distance to the directrix if the point be without the parabola, and less if it be within.

Let R [Fig. 18] be a point without the curve. Draw DR perpendicular to EG, and produce it to meet the curve. Join the points F and R, F and M ; then is

$$FR > RD.$$

For we have $FR+RM > FM > MD$;

and hence, $FR > MD-RM > RD$.

Again : Let R′ be a point within the curve. Draw R′M′D′ perpendicular to EG, and join the points F and M′, F and R′ : then is

$$FR' < R'D'.$$

For we have $FR' < FM'+M'R' < M'D'+M'R' < R'D'$.

Cor. Conversely, a point is within or without the curve, according as its distance from the focus is less or greater than its distance from the directrix.

PROPOSITION IV.

THEOREM.

A straight line bisecting the angle formed by two straight lines drawn from any point in the curve, the one to the focus, the other perpendicular to the directrix, is a tangent to the parabola at that point.

Let MD [Fig. 19] be drawn perpendicular to EG, and join M and F; then if the straight line MT bisect the angle DMF, it is a tangent to the parabola at M.

For, assume any other point, as N, in MT; draw ND′ perpendicular to EG, and join the points F and D, N and D, N and F.

Then, because MT bisects the vertical angle of the isosceles triangle FMD, it bisects the base FD at right angles; hence the triangles FNI and DNI are equal, and NF=ND.

But, ND is > ND′; therefore
NF is > ND′, and (Prop. III. cor.)
the point N is without the curve.

Thus it may be shown that every point of MT, except M, is without the curve.

COR. 1. From the above may be derived a method of drawing a tangent to the parabola, at any point in the curve; and also from a given point without the curve.

Let N be the given point, and from it as a centre, and with NF as a radius, describe an arc of a circle, cutting the directrix in D; draw DX′ parallel to the axis. M, the point in which it intersects the curve, is the point of contact, and the line joining M and N is the tangent required.

For, by construction, FN = DN; and since M is a point of the curve, FM = DM, and hence (Geom. I. 16, cor.) NT is perpendicular to FD at its middle point I, and (Geom. I. 11, schol.) bisects the angle FMD.

The same circumference cuts the directrix in another point, through which, if a line be drawn parallel to the axis, the point of contact of another tangent passing through N will be determined.

Cor. II. The angle FMD increases as M moves toward A, and at A becomes equal to two right angles; hence, the tangent at the vertex of the parabola is perpendicular to the axis.

Cor. III. The tangent at any point makes equal angles with the diameter at that point, and with the line drawn thence to the focus.

Cor. IV. The normal bisects the angle made by the diameter at the point of contact, with the line drawn from that point to the focus.

Let the normal MQ be drawn; then since

X′MN = IMD = IMF, we have QMX′ = QMF,

they being respectively complements of the equal angles X′MN and IMF.

Cor. V. If a tangent be produced to meet the axis, the points of intersection and contact are equally distant from the focus.

For, MTF = TMD;
also, FMT = TMD;
hence MTF = FMT,
and consequently FM = FT.

PROPOSITION V.

THEOREM.

If, from the focus, a straight line be drawn, meeting at right angles any tangent to the parabola, the point of intersection will be in the vertical tangent.

MT [Fig. 19] being any tangent to the parabola, the point I in which the perpendicular FI intersects it, lies in AA′ the tangent at the vertex.

It has been shown that the line FD is perpendicular to the tangent MT, and bisected by it; hence I is the middle point of FD.

Again: Since AA′ the tangent at the vertex is (Prop. IV. Cor. 2) parallel to the directrix, it divides the sides FB and FD of the triangle FBD proportionally; but it bisects FB; it must therefore bisect FD, and consequently meet it in the point I.

Cor. The perpendiculars let fall from the focus upon tangents to the parabola, are to each other as the square roots of the corresponding radii vectores.

Since AI is perpendicular to the hypothenuse of the triangle FIT, and passes through the vertex of the right angle at I, we have (Geom. IV. 23)

$$FA : FI :: FI : FT,$$

and hence $FA \times FT = FI^2$;

but (Prop. IV. Cor. 5) $FT = FM$;

therefore, $FA \times FM = FI^2$.

Let M′I′ be a tangent at any point of the curve, as M′, and FI′ the perpendicular drawn to it from the focus; then we shall also have $FA \times FM' = FI'^2$.

From these two equations we derive the proportion

$$FI^2 : FI'^2 :: FM : FM',$$

or

$$FI : FI' :: \sqrt{FM} : \sqrt{FM'}.$$

PROPOSITION VI.

THEOREM.

The radius vector of any point of the parabola, is greater than the corresponding abscissa by one-fourth the parameter.

FM [Fig. 19] being the radius vector of any point M of the curve, we have

$$FM = AP+AB = x+\frac{p}{4}.$$

For, FM=MD=PB=AP+AB; but (Def. IV.) AB $=\frac{p}{4}$;

hence,

$$FM = AP+\frac{p}{4}$$

$$= x+\frac{p}{4}.$$

PROPOSITION VII.

THEOREM.

The subnormal is the same for every point of the parabola, and equal to half the parameter.

MQ [Fig. 19] being a normal at any point M, the subnormal PQ $= \frac{1}{2}p$.

Since MQ and FI are perpendicular to the same straight line MT, they are parallel to each other; and hence

$$MQP = DFB.$$

Also the angles QPM and FBD are right angles, and PM = BD. The triangles MQP and DFB are therefore equal, and

$$PQ = FB.$$

But $FB = 2\,AB = \frac{4\,AB}{2} = \frac{p}{2}$; (Def. IV.)

hence, $PQ = \frac{1}{2}p.$

Cor. I. The triangle PMQ gives

$$MQ^2 = MP^2 + PQ^2,$$

or $MQ^2 = y^2 + \frac{p^2}{4} = px + \frac{p^2}{4}$, (Prop. II.)

and hence the *normal* $MQ = \sqrt{px + \frac{p^2}{4}}.$

Cor. II. $AQ = AP + PQ = x + \frac{p}{2}.$

PROPOSITION VIII.

THEOREM.

The subtangent for any point of the parabola is double the corresponding abscissa.

The subtangent PT = 2 AP.

For, the triangle TMQ [Fig. 19] being right-angled at M, and MP being perpendicular to the hypothenuse, we have (Geom. IV. 23)

$$PQ : PM :: PM : PT,$$

and hence $PT = \frac{PM^2}{PQ}$;

but (Prop. II.) $PM^2 = p\,.\,AP$,

and (Prop. VII.) $PQ = \frac{p}{2}$; and

hence, by substitution, $PT = \frac{p\,.\,AP}{\frac{1}{2}p} = 2\,AP.$

Cor. I. Hence the following method of drawing a tangent to the parabola, at any point M :

Draw the ordinate MP to the axis, and make AT = AP; the straight line drawn through the points T and M is the tangent required.

Cor. II. The triangle MPT = the rectangle MPAA′; for they have the common base MP, and the height of the triangle is double that of the parallelogram.

Cor. III. If we consider the tangent as limited by its intersection with the axis and the point of contact, then

$$\begin{aligned}\text{the tangent MT} &= \sqrt{\mathrm{MP}^2+\mathrm{PT}^2}\\ &= \sqrt{p.\mathrm{AP}+4\,\mathrm{AP}^2}\\ &= \sqrt{px+4\,x^2}.\end{aligned}$$

OF DIAMETERS.

Def. V. A straight line drawn from any point of the curve, parallel to the axis, is called a *diameter*. The point in the curve from which it is drawn, is called the *vertex* of that diameter.

Thus [Fig. 17], MX′, parallel to BX, is a diameter; and M is its vertex.

VI. An ordinate to a diameter, is a straight line drawn from any point in the curve, to meet that diameter, parallel to the tangent at its vertex.

Thus, M′P″, parallel to the tangent at M, is an ordinate to MX′; and MP″ is the corresponding abscissa.

VII. The parameter of a diameter, is a straight line equal to four times the distance from the vertex of that diameter to the directrix. It is denoted by p'.

PROPOSITION IX.

THEOREM.

The parameter of any diameter is greater than the parameter of the axis, by four times the abscissa of the vertex of that diameter.

Let MX′ [Fig. 19] be any diameter; then

its parameter $p' = p+4x$.

For (Def. VII.), $p' = 4\,\text{MD} = 4\,\text{PB} = 4\,\text{AB}+4\,\text{AP}$;

but, $4\,\text{AB} = p$;

then, $p' = p+4\,\text{AP} = p+4x$.

PROPOSITION X.

THEOREM.

If, from the extremities of a double ordinate to any diameter, ordinates be drawn to the axis, and also a tangent be drawn to the curve at the vertex of the diameter; then each of the triangles formed by the mutual intersection of the diameter and the ordinates produced if necessary, is equivalent to the parallelogram resulting from the intersection of the axis and diameter by the tangent and double ordinate.

Let M′M″ [Fig. 20] be a double ordinate to the diameter MX′, and let the ordinate M′P′ to the axis, and the tangent MT at M the vertex of the diameter, be drawn; then will the triangle M′KO be equivalent to the parallelogram MOST.

For (Prop. II. Cor. 1),

$$\begin{aligned}\text{MP}^2 : \text{M}'\text{P}'^2 &:: \text{AP} : \text{AP}' \\ &:: \text{AP}\times\text{PM} : \text{AP}'\times\text{PM} \\ &:: \text{APMR} : \text{AP}'\text{KR}.\end{aligned}$$

But the triangles MPT, M′P′S, similar because their sides are parallel, give (Geom. IV. 25)

$$MP^2 : M'P'^2 :: MPT : M'P'S;$$

hence, $MPT : M'P'S :: APMR : AP'KR;$

but (Prop. VIII. Cor. 2), $MPT = APMR;$

therefore, $M'P'S = AP'KR.$

But $M'P'S = KOSP' + M'KO,$

and $AP'KR = KOSP' + OSAR;$

hence, $M'KO = OSAR.$ .... [B]

Now the triangles YAT and MYR are equal, since they are right-angled and have the angles at Y equal, and the side AT = AP = the side MR; and hence

$$\begin{aligned} OSAR &= OSAYM + MYR \\ &= OSAYM + YAT \\ &= MOST. \quad \text{.... ....} \ [C] \end{aligned}$$

Therefore (Equation [B]),

$$M'KO = MOST.$$

Again: From M″ let the ordinate M″P″ be drawn to the axis, and produced to meet the diameter in K′; then will the triangle M″K′O be equivalent to the parallelogram MOST.

For, in the same way as in the first case, only using the letters M″, P″ and K′, instead of M′, P′ and K, it may be shown that $M''P''S = AP''K'R;$

and adding to each member of the equation the trapezoid OSP″K′, we have $M''K'O = OSAR;$

but (Equation [C]), $OSAR = MOST,$

hence $M''K'O = MOST.$

Cor. Every diameter bisects its double ordinates.

For, the triangles M′KO and M″K′O, being each equivalent to the parallelogram MOST, are equivalent to each other: they are moreover similar; consequently all their parts are equal, and $M'O = M''O.$

PROPOSITION XI.

THEOREM.

The square of an ordinate to a diameter, is equal to the product of the corresponding abscissa and the parameter of the diameter.

Denoting by p' the parameter of the diameter MX' [Fig. 20], $\quad M'O^2 = p'.MO.$

For, the triangles M'KO and MPT, similar because their sides are parallel, give

$$M'O^2 : MT^2 :: M'KO : MPT;$$

but (Prop. X.) $\quad M'KO = MOST,$ and

(Prop. VIII. Cor. 2) $\quad MPT = MPAR;$

hence, $\quad M'O^2 : MT^2 :: MOST : MPAR. \ldots. [D]$

Now the parallelograms MOST, MPAR, being of the same altitude MP, are to each other as their bases ST (or MO) and AP, and give

$$MO : AP :: MOST : MPAR;$$

therefore, comparing this with Proportion [D],

$$M'O^2 : MT^2 :: MO : AP.$$

But, $MT^2 = p.AP + 4\,AP^2;$ (Prop. VIII. Cor. 3.)

hence the above proportion becomes

$$M'O^2 : p.AP+4\,AP^2 :: MO : AP,$$

and

$$M'O^2 = \frac{p.AP+4\,AP^2}{AP} \times MO$$

$$= (p+4\,AP)\,MO:$$

but (Prop. IX.), $p+4\,AP = p';$ therefore,

$$M'O^2 = p'.MO.$$

Cor. I. $M''O^2 = p'.MO$; for $M''O = M'O$.

Cor. II. The squares of ordinates to a diameter are to each other as the corresponding abscissas.

CHAPTER III.

THE HYPERBOLA.

DEFINITIONS.

I. THE HYPERBOLA is a curve in which the difference of the distances of each point from two fixed points called the *foci*, is equal to a given line.

Let F and F′ [Fig. 21] be the foci of an hyperbola MAM′; then if from any point M of the curve the straight lines MF and MF′ be drawn, their difference MF′—MF is equal to a given line.

Remark. It will be shown that the hyperbola consists of two equal and opposite branches MAM′, M″A′M‴.

II. The middle of the straight line which joins the foci, is called the *centre* of the hyperbola. That part of it which is intercepted by the curve, is called the *transverse axis.*

Thus, C, the middle point of the straight line FF′, is the centre, and AA′ is the transverse axis of the hyperbola MAM′.

III. The line drawn through the centre, perpendicular to the transverse axis, and terminated by the circumference described from one of the extremities of the transverse axis as a centre, and with the distance from the centre to one of the foci as a radius, is called the *conjugate axis.*

Thus, the straight line BB′ drawn through C perpendicular to AA′, and terminated by the circumference described from A as a centre, and with CF as a radius, is the conjugate axis of the hyperbola MAM′.

IV. The *parameter* is a third proportional to the transverse and conjugate axes.

PROPOSITION I.

PROBLEM.

To describe an hyperbola.

FIRST METHOD : BY POINTS.

Let F and F′ [Fig. 21] be the foci, and X the given line.

Through F and F′ draw an indefinite straight line, and from C the middle of FF′ lay off CA and CA′ each equal to the half of X ; then A and A′ are points of the curve.

For, by construction,

$$CF-CA = CF'-CA', \quad \text{or } FA = F'A';$$

hence $F'A-FA = F'A-F'A' = AA' = X,$

and $FA'-F'A' = FA'-FA = AA' = X.$

That is, the difference of the distances of each of the points A and A′ from the foci is equal to the given line, and hence (Def. I.) they are points of the hyperbola.

Remark. In the ellipse, the points A and A′ lie without F and F′ ; in the hyperbola, they lie within : so that

in the ellipse we have $X > FF'$;

in the hyperbola, $X < FF'$.

To find other points of the curve : On the line CF produced, and to the right of the point F, take any point L, and from F and F′ as centres, and with AL and A′L as radii, describe two arcs of circles; the points M and M′ in which they intersect, are points of the curve.

For, drawing the lines FM, F′M, FM′, F′M′, we have

$$F'M - FM = A'L - AL = AA' = X,$$

and

$$F'M' - FM' = A'L - AL = AA' = X.$$

Reciprocally, if with AL and A′L as radii, and from F′ and F as the respective centres, arcs be described, two other points M″ and M‴ will be determined.

By giving to L different positions, other points may in like manner be determined.

Remark. The preceding construction requires the point L to be situated to the right of F; for were it on the other side, at L′ for instance, we should have

$$AL' < AF,$$

or

$$AA' + 2\,AL' < AA' + 2\,AF,$$

or

$$A'L' + AL' < FF';$$

that is, the sum of the radii A′L′, AL′ less than FF′ the distance of the centres, so that the circumferences would not meet.

Cor. I. If through C a straight line BB′ be drawn perpendicular to FF′, it is evident that in relation to this line the points M″ and M‴ are situated in precisely the same manner as the points M and M′; so that for any two points of the curve to the right of BB′, there are two corresponding points to the left of that line. Hence the hyperbola is composed of two equal and opposite branches MAM′, M″A′M‴, similarly situated with respect to BB′. Moreover, as we may take L as far to the right of F on CF prolonged as we please, or, in other words, may employ radii indefinitely great, these two branches are infinite in extent.

Cor. II. It may also be readily shown that the line MM′ is perpendicular to FF′, and bisected by it : consequently the curve is divided by the line FF′ into two equal parts, similarly situated with regard to it.

SECOND METHOD : BY CONTINUOUS MOTION.

Let one of the extremities of a ruler F′G, greater than FF′, be fixed at the point F′, and at the other extremity let a thread be attached of such a length that the length of the ruler shall exceed that of the thread by the constant quantity X; also let the other extremity of the thread be fixed at the point F; then move the ruler about the point F′, at the same time keeping the thread tense by applying a portion of it to the ruler by means of a pencil : the point of the pencil will describe a portion of an hyperbola. For in every position of the ruler, we shall have

$$(F'M+MG)-(MF+MG) = X,$$

or

$$F'M-FM = X.$$

PROPOSITION II.

THEOREM.

The semi-conjugate axis is a mean proportional between the distances from either of the foci to the extremities of the transverse axis.

That is [Fig. 22],

$$FA : BC :: BC : FA'.$$

For, $BC^2 = AB^2-AC^2 = CF^2-CA^2$, (Def. III.)

or $BC^2 = (CF+CA)\times(CF-CA) = FA\times FA'$;

and hence, $FA : BC :: BC : FA'$.

PROPOSITION III.

THEOREM.

The parameter is greater than four times the distance from one of the vertices of the transverse axis, to the adjacent focus.

That is, the parameter being denoted by p, we have

$$p > 4\,\mathrm{AF}. \qquad [\text{Fig. } 22.]$$

For (Def. IV.)

$$2\,\mathrm{AC} : 2\,\mathrm{BC} :: 2\,\mathrm{BC} : p,$$

or

$$p = \frac{2\,\mathrm{BC}^2}{\mathrm{AC}}.$$

But (Prop. II.),

$$\mathrm{BC}^2 = \mathrm{FA} \times \mathrm{FA}' = \mathrm{FA}(2\,\mathrm{AC} + \mathrm{FA});$$

hence, by substitution, we get

$$p = \frac{4\,\mathrm{FA} \times \mathrm{AC} + 2\,\mathrm{FA}^2}{\mathrm{AC}}$$

$$= 4\,\mathrm{FA} + \frac{2\,\mathrm{FA}^2}{\mathrm{AC}}.$$

PROPOSITION IV.

THEOREM.

The square of an ordinate to the transverse axis, is to the product of the corresponding abscissas, as the square of the conjugate axis, is to the square of the transverse axis.

That is, if MP [Fig. 22] be an ordinate to the transverse axis; then,

$$\mathrm{MP}^2 : \mathrm{AP} \times \mathrm{A'P} :: \mathrm{BB'}^2 : \mathrm{AA'}^2.$$

Let AC $= a$, BC $= b$, F′C $=$ FC $= c$;
and let CP $= x$, and MP $= y$: then,
PF $= c-x$, and PF′ $= c+x$.

Also let F′M $= z+a$, z being a variable quantity; then, FM $= z-a$.

The right-angled triangles PMF, PMF′ give

$$\mathrm{FM}^2 = \mathrm{MP}^2+\mathrm{PF}^2,$$

and

$$\mathrm{F'M}^2 = \mathrm{MP}^2+\mathrm{PF'}^2;$$

or, substituting,

$$(z-a)^2 = y^2+(c-x)^2,$$

and

$$(z+a)^2 = y^2+(c+x)^2.$$

Developing, and subtracting the first equation from the second, we find

$$z = \frac{cx}{a};$$

and substituting this value of z in the second equation, we have

$$\left(\frac{cx}{a}+a\right)^2 = y^2+(c+x)^2,$$

or

$$c^2x^2+2a^2cx+a^4 = a^2y^2+a^2c^2+2a^2cx+a^2x^2,$$

or

$$c^2x^2-a^2x^2-a^2c^2+a^4 = a^2y^2,$$

or

$$x^2(c^2-a^2)-a^2(c^2-a^2) = a^2y^2;$$

or, since $c^2-a^2 = b^2$,

$$(x^2-a^2)b^2 = a^2y^2. \quad\text{[A]}$$

And hence we derive the proportion,

$$y^2 : x^2 - a^2 :: b^2 : a^2;$$

that is,

$$\mathrm{MP}^2 : \mathrm{CP}^2-\mathrm{CA}^2 :: \mathrm{CB}^2 : \mathrm{CA}^2, \quad\text{[B]}$$

or

$$\mathrm{MP}^2 : \mathrm{CP}^2-\mathrm{CA}^2 :: \mathrm{BB'}^2 : \mathrm{AA'}^2.$$

But, $\mathrm{CP}^2-\mathrm{CA}^2 = (\mathrm{CP}+\mathrm{CA})\times(\mathrm{CP}-\mathrm{CA}) = \mathrm{AP}\times\mathrm{A'P}$;
hence,

$$\mathrm{MP}^2 : \mathrm{AP}\times\mathrm{A'P} :: \mathrm{BB'}^2 : \mathrm{AA'}^2.$$

Cor. I. The squares of ordinates to the transverse axis are to each other as the products of the corresponding abscissas.

For, M′P being any other ordinate to the transverse axis,

we have $MP^2 : AP \times A'P :: BB'^2 : AA'^2$,

and $M'P'^2 : AP' \times A'P' :: BB'^2 : AA'^2$; and

hence, $MP^2 : M'P'^2 :: AP \times A'P : AP' \times A'P'$.

Cor. II. $MP^2 : M'P'^2 :: CP^2 - CA^2 : CP'^2 - CA^2$.

Cor. III. Since $p = \frac{2\,BC^2}{AC}$, we have, by multiplying and dividing the second member by AC,

$$p = \frac{2\,BC^2.AC}{AC^2};$$

and hence, $p : 2\,AC :: BC^2 : AC^2$;

or (Proportion [B])

$$p : 2\,AC :: MP^2 : CP^2 - CA^2, \text{ and}$$

consequently $MP^2 = (CP^2 - CA^2)\frac{p}{2\,AC}$.

PROPOSITION V.

THEOREM.

The square of an ordinate to the conjugate axis, is to the sum of the squares of the corresponding abscissa and the semi-conjugate axis, as the square of the transverse axis is to the square of the conjugate axis.

That is, if MP″ [Fig. 22] be an ordinate to the axis BB′; then,

$$MP''^2 : CP''^2 + CB^2 :: AA'^2 : BB'^2.$$

Let $CP'' = x'$, and $MP'' = y'$; then,

$x = CP = MP'' = y'$, and $y = MP = CP'' = x'$.

Substituting these values of x and y in Equation [A],

$$a^2y^2 = (x^2-a^2)b^2,$$

we have $$a^2x'^2 = (y'^2-a^2)b^2;$$

and multiplying and transposing,

$$b^2y'^2 = a^2x'^2+a^2b^2 = a^2(x'^2+b^2),$$

which gives $$y'^2 : x'^2+b^2 :: a^2 : b^2$$
$$:: (2a)^2 : (2b)^2;$$

that is, $$\mathrm{MP}''^2 : \mathrm{CP}''^2+\mathrm{CB}^2 :: \mathrm{AA}'^2 : \mathrm{BB}'^2.$$

PROPOSITION VI.

THEOREM.

The double ordinate to the transverse axis, drawn through either focus of an hyperbola, is equal to the parameter.

That is, if NN′ [Fig. 22] be the double ordinate to AA′ drawn through one of the foci, as F; then,

$$\mathrm{AA}' : \mathrm{BB}' :: \mathrm{BB}' : \mathrm{NN}'.$$

We have (Proposition IV. Proportion [B]),

$$\mathrm{MP}^2 : \mathrm{CP}^2-\mathrm{AC}^2 :: \mathrm{BC}^2 : \mathrm{AC}^2;$$

and consequently when the abscissa CP becomes CF, we have $$\mathrm{FN}^2 : \mathrm{CF}^2-\mathrm{AC}^2 :: \mathrm{BC}^2 : \mathrm{AC}^2;$$

but (Def. III.) $$\mathrm{CF}^2-\mathrm{AC}^2 = \mathrm{AB}^2-\mathrm{AC}^2 = \mathrm{BC}^2;$$

hence $$\mathrm{FN}^2 : \mathrm{BC}^2 :: \mathrm{BC}^2 : \mathrm{AC}^2,$$

and $\mathrm{FN} : \mathrm{BC} :: \mathrm{BC} : \mathrm{AC}$; or inverting the proportion, and doubling each term,

$$\mathrm{AA}' : \mathrm{BB}' :: \mathrm{BB}' : \mathrm{NN}'.$$

PROPOSITION VII.

THEOREM.

The difference of the lines drawn from the foci to a point in the plane of the hyperbola, is less than the transverse axis if the point be without the curve, but greater if it be within.

That is, if R′ [Fig. 22] be a point without the hyperbola, and R a point within the curve; then we have,

1. $F'R' - FR' < AA'$; and,
2. $F'R - FR > AA'$.

For, in the first case we have

$F'R' = MF' - MR'$ and $FR' > MF - MR'$; and consequently,

$$\begin{aligned} F'R' - FR' &< MF' - MR' - MF + MR' \\ &< MF' - MF \\ &< AA'. \end{aligned}$$

And in the second case we have

$F'R = MF' + MR$, and $FR < MF + MR$; and consequently,

$$\begin{aligned} F'R - FR &> MF' + MR - MF - MR \\ &> MF' - MF \\ &> AA'. \end{aligned}$$

Cor. A point is without or within the hyperbola, according as the difference of the lines drawn from it to the foci is less or greater than the transverse axis.

PROPOSITION VIII.

THEOREM.

The straight line which bisects the angle contained by the radii vectores of any point of the hyperbola, is a tangent to the curve at that point.

That is [Fig. 23], M being any point in the curve, if the straight line DMN be drawn so as to bisect the angle FMF′; then will every point of that line, except M, be without the curve, and consequently DMN will be a tangent to the curve at M.

Let D be any point except M of the line MN; make MI = MF, and draw the straight lines FI, DI, DF and DF′.

DN, which bisects the angle FMI of the isosceles triangle FMI, bisects the base FI at right angles; hence the triangle FDI is isosceles, and we have

$$FD = DI;$$

therefore, $F'D - FD = F'D - DI$:

now $F'D$ is $< DI + F'I$, and hence

$$F'D - DI \text{ is } < F'I < AA' \quad (\text{since } F'I = AA');$$

consequently $F'D - FD < AA'$,

and therefore the point D is without the curve.

Cor. I. From the above, a method of drawing a tangent to the hyperbola, at any point M of the curve, is obvious.

Cor. II. If it be required to draw a tangent to the hyperbola, which shall pass through a given point D without the curve; then from F′ and D as centres, and with radii equal respectively to AA′ and FD, describe two arcs of circles; and through I their point of intersection and the focus F′, draw F′I meeting the curve in M; NMD is the tangent required.

For, since F′I = AA′, and F′M−FM = AA′,
we have F′I = F′M−FM,
or F′I = F′I+MI−FM,
and hence MI = MF.
We also have DF = DI.

Hence it may be easily shown that DN bisects the angle FMI: it is therefore a tangent to the curve at M.

As the circles intersect in two points, two tangents may be drawn to the curve from the given point D.

Cor. III. The opposite vertical angles DMI′, NMI are equal; also NMI = NMF:
consequently DMI′ = NMF.

Cor. IV. The tangents at the extremities of the transverse axis are perpendicular to that axis.

PROPOSITION IX.

THEOREM.

If, from the foci, perpendiculars be drawn to a tangent at any point of the hyperbola, they will meet it in the circumference of the circle described on the transverse axis.

That is, if from F and F′ [Fig. 24] perpendiculars be drawn to NN′, the points N and N′ are in the circumference of the circle described on AA′.

In addition to the preceding construction, draw F′N′ perpendicular to NN′, join the points C and N, and produce CN to meet F′N′ in D.

FN is perpendicular to NN′ by the preceding theorem. Since FC = CF′ and FN = NI, CN is parallel to F′I (Geom. IV. 16), and the triangles F′FI and CFN are similar; therefore,

$$CN = \tfrac{1}{2}\,F'I = \tfrac{1}{2}\,AA' = AC.$$

Again, since DN and F′I are parallel to each other, and also FN and F′N′, they being perpendicular to the same straight line NN′, the figure DNIF′ is a parallelogram, and

$$DN = F'I = 2\,AC;$$

but $$CN = AC;$$

hence $$DN - CN = AC,$$

or $$DC = AC.$$

Hence if from C as a centre, and with CA as a radius, the circumference of a circle be described, it will pass through the points N and D; and since NN′D is a right angle, it will also pass through N′.

PROPOSITION X.

THEOREM.

If, from the foci of an hyperbola, perpendiculars be drawn to a tangent at any point of the curve, then will the product of the perpendiculars be equal to the square of the semi-conjugate axis.

That is, the construction being the same as in the preceding proposition [Fig. 24],

$$FN \times F'N' = BC^2.$$

For (Geom. IV. 29),

$$F'N' \times F'D = F'A' \times F'A;$$

but since DNIF′ is a parallelogram, F′D = FN, and
(Prop. II.) $F'A' \times F'A = BC^2$;
hence, $F'N' \times FN = BC^2$.

Cor. In addition to the preceding construction, let FN″ be drawn perpendicularly to the tangent at M′; then,

$$FN : FN'' :: \sqrt{\frac{FM}{F'M}} : \sqrt{\frac{FM'}{F'M'}}.$$

For, the right-angled triangles FNM, F′N′M, having the angles FMN, F′MN′ equal, are similar; and hence,

$$FN : FM :: F'N' : F'M,$$

and
$$FN = \frac{FM \times F'N'}{F'M};$$

or, multiplying each member of the equation by FN,

$$FN^2 = FN \times F'N' \times \frac{FM}{F'M} = BC^2 \times \frac{FM}{F'M},$$

and
$$FN = BC.\sqrt{\frac{FM}{F'M}},$$

hence, for the perpendicular FN″ upon the tangent at M′,

$$FN'' = BC.\sqrt{\frac{FM'}{F'M'}},$$

and consequently
$$FN : FN'' :: \sqrt{\frac{FM}{F'M}} : \sqrt{\frac{FM'}{F'M'}}.$$

Scholium. This is the same result apparently as in the ellipse (Chap. I. Prop. XII.) : but in that curve, since $FM+F'M = AA'$, as FM increases, F′M diminishes; whilst in the hyperbola, since $F'M-FM = AA'$, or $F'M=AA'+FM$, as FM increases, F′M also increases.

PROPOSITION XI.

PROBLEM.

To find the algebraic expression of the radii vectores of the hyperbola.

From the foci, draw FM, F′M [Fig. 22] to any point M of the hyperbola : Required the algebraic expressions of these lines.

Employing the same algebraic symbols as in the fourth proposition, we have F′M $= z+a$ and FM $= z-a$, and it is there shown that $z = \frac{cx}{a}$; hence, by substitution,

$$\text{F}'\text{M} = \frac{cx}{a} + a = \frac{cx+a^2}{a},$$

and

$$\text{FM} = \frac{cx}{a} - a = \frac{cx-a^2}{a}.$$

PROPOSITION XII.

PROBLEM.

To find the algebraic expression of the subnormal of the hyperbola.

Draw MQ [Fig. 23] perpendicular to the tangent DN at M the point of contact; then MQ is the normal to the hyperbola at M, and PQ is the subnormal : Required the algebraic expression of the latter line.

The lines FN and QM, being perpendicular to the tangent DN, are therefore parallel to each other, and we have

$$\text{F}'\text{I} : \text{F}'\text{F} :: \text{IM} : \text{FQ},$$

or $$\text{F}'\text{I} : \text{F}'\text{F} :: \text{FM} : \text{FQ},$$

or $$2a : 2c :: \frac{cx-a^2}{a} : \text{FQ};$$

hence, $$\text{FQ} = 2c \,.\, \frac{cx-a^2}{2a^2} = \frac{c^2x-ca^2}{a^2}.$$

Also, $\text{FP} = \text{CF}-\text{CP} = c-x$; but the subnormal

$$\text{PQ} = \text{FQ}+\text{FP} = \frac{c^2x-ca^2}{a^2}+c-x$$

$$= \frac{c^2x-ca^2+ca^2-xa^2}{a^2}$$

$$= \frac{c^2x-a^2x}{a^2} = \frac{(c^2-a^2)x}{a^2} = \frac{b^2}{a^2}x.$$

Cor. Denoting by p the parameter to the transverse axis, we have

$$2a : 2b :: 2b : p;$$

hence, $$p = \frac{4b^2}{2a}, \quad \text{and} \quad \frac{p}{2a} = \frac{b^2}{a^2};$$

therefore, $$\text{PQ} = \frac{px}{2a}, \quad \text{and} \quad \text{PQ}.2a = px;$$

whence $$2a : p :: x : \text{PQ},$$

or $$\text{AA}' : p :: \text{PC} : \text{PQ},$$

and hence (Geom. IV. 2) PQ may be determined.

Remark. The method of drawing a normal or tangent to the curve, by means of the subnormal, is obvious.

PROPOSITION XIII.

PROBLEM.

To find the algebraic expression of the subtangent of the hyperbola.

Since PM [Fig. 23] is perpendicular to TQ the hypothenuse of the right-angled triangle TMQ, we have

$$\mathrm{PQ} : \mathrm{PM} :: \mathrm{PM} : \mathrm{PT},$$

or

$$\frac{b^2}{a^2}x : y :: y : \mathrm{PT};$$

and hence

$$\mathrm{PT} = \frac{y^2}{\frac{b^2}{a^2}x};$$

but

$$y^2 = \frac{b^2}{a^2}(x^2-a^2); \quad \text{(Prop. IV. [A])}$$

hence,

$$\mathrm{PT} = \frac{\frac{b^2}{a^2}(x^2-a^2)}{\frac{b^2}{a^2}x} = \frac{x^2-a^2}{x}$$

Cor. I.

$$\mathrm{PT} = \frac{x^2-a^2}{x} = \frac{(x+a).(x-a)}{x};$$

hence

$$x : x+a :: x-a : \mathrm{PT}$$

that is,

$$\mathrm{CP} : \mathrm{A'P} :: \mathrm{AP} : \mathrm{PT},$$

and hence (Geom. IV. 2) PT may be determined.

Remark. The method of drawing a tangent to the hyperbola, by means of the subtangent, is obvious.

Cor. II. If on AA′ as a transverse axis, different hyperbolas be described, and at the points which have a common abscissa x, tangents be drawn, they will intersect AA′, produced if necessary, in the same point; for the expression of the subtangent, $\frac{x^2-a^2}{x}$, is independent of the conjugate axis.

Cor. III. If a tangent to the hyperbola meet the transverse axis, and from the point of contact an ordinate be drawn to the same axis; then is the semi-transverse axis a mean proportional between the distances from the centre to the intersections of the axis by the ordinate and tangent. That is,

$$CP : CA :: CA : CT.$$

For, denoting the subtangent by s, we have

$$sx = x^2-a^2,$$

or

$$a^2 = x^2-sx = x(x-s);$$

and hence, $x : a :: a : x-s;$

that is, $CP : CA :: CA : CT.$

Cor. IV. The equation $PT = \frac{x^2-a^2}{x}$ gives

$$PT \times PC = CP^2-CA^2.$$

PROPOSITION XIV.

PROBLEM.

To find the algebraic expression of the subtangent, measured on the conjugate axis.

The similar triangles MPT, MP′T′ [Fig. 23] give

$$TP : PM :: MP'=CP : P'T',$$

or $$\frac{x^2-a^2}{x} : y :: x : \text{P}'\text{T}' :$$

hence $$\text{P}'\text{T}' = \frac{yx}{\frac{x^2-a^2}{x}} = \frac{yx^2}{x^2-a^2};$$

but $a^2y^2 = b^2(x^2-a^2)$; hence $x^2-a^2 = \dfrac{a^2y^2}{b^2}$,

and $x^2 = \dfrac{a^2y^2}{b^2}+a^2 = \dfrac{a^2y^2+a^2b^2}{b^2}$.

Substituting these values of x^2 and x^2-a^2 in the expression for P′T′, we have

$$\text{P}'\text{T}' = \frac{y.\frac{a^2y^2+a^2b^2}{b^2}}{\frac{a^2y^2}{b^2}} = \frac{y(a^2y^2+a^2b^2)}{a^2y^2}$$

$$= \frac{b^2+y^2}{y}.$$

Cor. CP′ : CB :: CB : CT′.

For, denoting P′T′ by s', we have

$$s' = \frac{b^2+y^2}{y}, \text{ and } b^2 = s'y-y^2 = y(s'-y);$$

and hence $y : b :: b : s'-y$,

that is, CP′ : CB :: CB : CT′,

and thus CT′ can be determined.

OF DIAMETERS AND ASYMPTOTES.

DEF. V. The *asymptotes* of an hyperbola are the indefinite straight lines drawn through the centre of the curve, so as to cut the tangent at the vertex at a distance from the vertex equal to the semi-conjugate axis.

Thus, the straight lines ZY, Z′Y′ [Fig. 26], drawn through C, and cutting XX′ the tangent at A, so that AX and AX′ are each equal to BC, are the asymptotes of the hyperbola MAM″.

VI. Any straight line drawn through the centre, and terminated by the opposite branches of the hyperbola, is called a *diameter*.

Thus, MM′ is a diameter of the hyperbola MAM″.

VII. A straight line passing through the centre, parallel to the tangent at one of the extremities of a diameter, and terminated by the straight lines drawn from the point of contact parallel to the asymptotes, is called a *conjugate* to that diameter.

Thus, NN′ drawn through C parallel to TT′ the tangent at M, and terminated by the straight lines MN, MN′ drawn from M parallel to Z′Y′ and ZY, is the conjugate to MM′.

VIII. An *ordinate* to a diameter, is a straight line drawn from any point of the curve, parallel to the tangent at the extremity of that diameter, and terminating in the diameter produced. The parts of the diameter produced which are intercepted by its extremities and the foot of the ordinate, are called *abscissas*.

Thus, M″Q [Fig. 28] is an ordinate to the diameter MM′, and MQ and M′Q are the corresponding abscissas.

PROPOSITION XV.

THEOREM.

If a radius vector be drawn from the extremity of any diameter, to meet its conjugate, produced if necessary, the part intercepted by the conjugate is equal to the semi-transverse axis.

If the radius vector MF′ [Fig. 25] be drawn from M the extremity of any diameter MM′, the part MG′ intercepted by the conjugate NN′ is equal to AC.

For, drawing FD parallel to MT, we have the angles MDF, MFD equal respectively to their alternate angles T′MD, TMF; but (Prop. VIII. cor. 3) the latter angles are equal to each other: hence we have

$$MDF = MFD,$$

and consequently $MF = MD.$

Also since CG′ is parallel to FD, and F′C=FC, we have

$$F'G' = G'D;$$

therefore, $MG' = G'D - MD$

$= G'F' - MF.$

Also, $MG' = MF' - G'F'$; hence,

by addition, $2\,MG' = MF' - MF = 2\,AC,$

and $MG' = AC.$

Cor. The triangles MFD, MGG′ being similar, and MFD isosceles, we have MGG′ isosceles; and hence

$$MG = MG' = AC.$$

PROPOSITION XVI.

THEOREM.

If, from any point of an hyperbola, an ordinate be drawn to the transverse axis, and produced to meet the asymptotes; then will the rectangle of the segments into which it is divided by that point, be equal to the square of the semi-conjugate axis.

That is, if the ordinate DP [Fig. 26] to AA′ be produced, meeting the asymptotes in Z and Z′; then,

$$\mathrm{DZ}\times\mathrm{DZ'} = \mathrm{BC}^2.$$

The similar triangles CAX, CPZ give

$$\mathrm{CA} : \mathrm{AX} :: \mathrm{CP} : \mathrm{PZ},$$

or

$$a : b :: x : \mathrm{PZ};$$

and hence,

$$\mathrm{PZ} = \frac{bx}{a}.$$

Now DZ = PZ−PD, and DZ′ = PZ′+PD : but PD is denoted by y, and the triangles CPZ, CPZ′ being equal, PZ = PZ′; hence we have, by substitution,

$$\mathrm{DZ} = \frac{bx}{a} - y,$$

and

$$\mathrm{DZ'} = \frac{bx}{a} + y,$$

and, by multiplication,

$$\mathrm{DZ}\times\mathrm{DZ'} = \frac{b^2x^2}{a^2} - y^2 :$$

but (Prop. IV. Equation [A]),

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2);$$

therefore $$DZ \times DZ' = \frac{b^2x^2}{a^2} - \frac{b^2x^2}{a^2} + b^2 = b^2,$$

or $$DZ \times DZ' = BC^2.$$

Cor. I. $$DZ : BC :: BC : DZ'.$$

Cor. II. As the asymptotes and the branches of the hyperbola are produced, the latter continually approach the former, but never meet them.

1. As the point D moves on the curve from the vertex A, DZ′ increases; but from the equation

$$DZ \times DZ' = BC^2,$$

it appears that as DZ′ increases, DZ diminishes : consequently the branches of the curve continually approach the asymptotes.

2. Since $PZ = \frac{bx}{a}$, or $PZ^2 = \frac{b^2x^2}{a^2}$, and (Prop. IV. Equa. [A]), $PD^2 = \frac{b^2x^2}{a^2} - b^2$, we always have $PZ > PD$; that is, no point D of the curve can ever meet the corresponding point Z of the asymptote.

Schol. If, in the equation $PD^2 = \frac{b^2x^2}{a^2} - b^2$, we suppose x infinite, then the finite term b^2 may be dropped, giving

$$PD^2 = \frac{b^2x^2}{a^2}, \text{ and consequently } PD = PZ.$$

We may say, then, that the branches of the hyperbola meet the asymptotes at an infinite distance from the centre.

PROPOSITION XVII.

THEOREM.

If, from any point of the curve, a line be drawn to meet one of the asymptotes, parallel to the other, the rectangle of this line and the distance from the centre to its intersection with the asymptote is constant, and equal to the square of half the diagonal of the rectangle described on the semi-axes.

That is, MO [Fig. 26] being drawn from any point M of the curve, parallel to CZ'; then,

$$MO.CO = AK^2.$$

Through M, let II' be drawn perpendicular to AA'. Since BC is equal and parallel to AX', we have AB parallel to CX', and hence parallel to MO; we also have AX parallel to MI : therefore the triangles AKX, MOI are similar, and

give $\quad MO : AK :: MI : AX$,

or $\quad :: MI : BC$:

but (Prop. XVI. cor. 1),

$$MI : BC :: BC : I'M;$$

therefore $\quad MO : AK :: BC : I'M$. .... [C]

Again, the similar triangles AKX, $I'I''M$ give

$$AX : I'M :: XK : MI'',$$

or $\quad BC : I'M :: XK : MI''$:

but [C], $\quad MO : AK :: BC : I'M$;

therefore, $\quad MO : AK :: XK : MI''$,

and $\quad MO \times MI'' = AK \times XK$;

or, since (Geom. I. 31) $AK = XK$ and $MI'' = CO$,

$$MO \times CO = AK^2.$$

Cor. I. Let MO $= t$, CO $= u$, and AK $= h$; then,

$$ut = h^2.$$

The quantity h^2 is called the *square* of the hyperbola.

Cor. II. The triangle ABC gives

$$AB^2 = AC^2 + BC^2;$$

and hence $$(2h)^2 = a^2 + b^2,$$

$$4h^2 = a^2 + b^2,$$

and $$h^2 = \frac{a^2}{4} + \frac{b^2}{4}.$$

Cor. III. Since $MO \times CO = AK^2$, and AK = CK, we have

$$MO \times CO = CK \times AK;$$

and hence, $$MO : AK :: CK : CO.$$

PROPOSITION XVIII.

THEOREM.

If a straight line be drawn, intersecting the hyperbola and meeting the asymptotes, the parts comprehended between the asymptotes and the curve will be equal to each other.

Let QQ′ [Fig. 27] be any straight line intersecting the curve in M′ and M″, and meeting the asymptotes in Q and Q′; then will Q′M″ = QM′.

Through M′ and M″ draw VV′ and RR′ perpendicular to the transverse axis; then the similar triangles R′Q′M″, V′Q′M′ give

$$Q'M'' : R'M'' :: Q'M' : V'M';$$

and the similar triangles $QM''R$, $QM'V$ give

$$QM'' : RM'' :: QM' : VM';$$

and multiplying these two proportions term by term, we have

$$Q'M''.QM'' : R'M''.RM'' :: Q'M'.QM' : V'M'.VM'.$$

But (Prop. XVI.) the second and fourth terms are equal;

hence $\quad Q'M''.QM'' = QM'.Q'M',$

or $\quad Q'M''(QM'+M'M'') = QM'(Q'M''+M'M''),$

or $\quad Q'M''.QM'+Q'M''.M'M'' = QM'.Q'M''+QM'.M'M'',$

or $\quad Q'M''.M'M'' = QM'.M'M'',$

or $\quad Q'M'' = QM'.$

COR. I. If a tangent be drawn to the hyperbola at any point, the part intercepted by the asymptotes is bisected at the point of contact.

For we may consider the tangent TT' as a secant in which the part within the curve is reduced to zero; and by the proposition, we have the exterior segments MT, MT' equal to each other.

COR. II. $\quad MO = ON. \quad$ [Fig. 26.]

The similar triangles $CT'T$ and OMT give

$$T'T : MT :: T'C : MO.$$

But, $MT = \frac{1}{2}TT'$; hence, $MO = \frac{1}{2}T'C = \frac{1}{2}MN$;

consequently $\quad MO = ON.$

PROPOSITION XIX.

THEOREM.

If a tangent to the hyperbola at any point be produced to meet the asymptotes, and straight lines be drawn parallel to it, intersecting the curve, and terminated by the asymptotes; then the rectangles of the segments into which these lines are divided by the curve are equal to each other, and equal each to the square of half the tangent.

That is, if QQ′, SS′ [Fig. 27] be drawn parallel to TT′; then,

$$Q'M'' \times QM'' = S'M^{iv} \times SM^{iv} = TM^2.$$

Through M″ and M^iv, let RR′, VV′ be drawn perpendicular to the transverse axis; then the similar triangles R′M″Q′, V′M^ivS′ give

$$Q'M'' : R'M'' :: S'M^{iv} : V'M^{iv};$$

and the similar triangles M″RQ, M^ivVS give

$$QM'' : RM'' :: SM^{iv} : VM^{iv};$$

and by multiplication, we have

$$Q'M''.QM'' : R'M''.RM'' :: S'M^{iv}.SM^{iv} : V'M^{iv}.VM^{iv}.$$

But (Prop. XVI.) the consequents in this proportion are equal; hence the antecedents are also equal, and we have

$$Q'M''.QM'' = S'M^{iv}.SM^{iv}.$$

Now if the secant QQ′ be supposed to coincide with the tangent, it is evident that the above equation will still be true, and we shall have

$$\begin{aligned} S'M^{iv} \times SM^{iv} &= T'M \times TM \\ &= TM^2. \text{ (Prop. XVIII. cor. 1.)} \end{aligned}$$

PROPOSITION XX.

THEOREM.

Every diameter, and its conjugate, are bisected at the centre.

Let MM′ [Fig. 26] be any diameter, and NN′ its conjugate; then, MC = M′C, and NC = N′C.

Take Cp = Cp′; draw the ordinates Mp and M′p′; and join C and M, C and M′: then we have p′M′ = pM; and hence the right-angled triangles CpM, Cp′M′ are equal to each other, and CM = CM′.

Also since the vertical angles pCM, p′CM′ are equal, and AA′ is a straight line, MC is in the same straight line with M′C; and MM′, bisected at C, is a diameter.

Again: The parallelogram MTCN′ gives CN′ = MT, and the parallelogram MNCT′ gives CN = MT′: but (Prop. XVIII. cor. 1) MT = MT′; hence

CN = CN′.

Cor. MN is parallel to M′N′.

PROPOSITION XXI.

THEOREM.

The square of an ordinate to any diameter, is to the rectangle of the corresponding abscissas, as the square of the conjugate to that diameter, is to the square of the diameter itself.

Let MM′ [Fig. 28] be any diameter, NN′ its conjugate, and M″Q any ordinate to MM′; then,

$$M''Q^2 : M'Q \times MQ :: NN'^2 : MM'^2.$$

The tangent MT being drawn, and the ordinate produced to meet the asymptotes, we have, by similar triangles,

$$\mathrm{CM} : \mathrm{MT} :: \mathrm{CQ} : \mathrm{QQ}';$$

or, putting $\mathrm{CM} = a'$, $\mathrm{MT} = \mathrm{CN} = b'$, and $\mathrm{CQ} = x'$,

$$a' : b' :: x' : \mathrm{QQ}';$$

and hence $$\mathrm{QQ}' = \frac{b'x'}{a'}.$$

Now since Q′Q″ is parallel to TT′, and $\mathrm{TM} = \mathrm{T'M}$, (Prop. XVIII. cor. 1) $\mathrm{QQ}'' = \mathrm{QQ}'$;

and moreover $\mathrm{Q''M'''} = \mathrm{Q'M''}$; (Prop. XVIII.)

therefore $\mathrm{QM''} = \mathrm{QM'''}$.

Denoting then QM″ or QM‴ by y', we have

$$\mathrm{Q'M'''} = \mathrm{QQ'} + \mathrm{QM'''} = \frac{b'}{a'}x' + y',$$

and $$\mathrm{Q''M'''} = \mathrm{QQ''} - \mathrm{QM'''} = \frac{b'}{a'}x' - y';$$

and hence, by multiplication,

$$\mathrm{Q'M'''} \times \mathrm{Q''M'''} = \frac{b'^2}{a'^2}x'^2 - y'^2.$$

But (Prop. XIX.) $\mathrm{Q'M'''} \times \mathrm{Q''M'''} = \mathrm{TM}^2 = b'^2$;

therefore $$\frac{b'^2}{a'^2}x'^2 - y'^2 = b'^2,$$

and $$b'^2x'^2 - a'^2y'^2 = a'^2b'^2,$$

or $$a'^2y'^2 = b'^2x'^2 - a'^2b'^2 = b'^2(x'^2 - a'^2);$$

hence $$y'^2 : x'^2 - a'^2 :: b'^2 : a'^2,$$

and $$y'^2 : (x'+a').(x'-a') :: b'^2 : a'^2;$$

that is, $$\mathrm{M''Q}^2 : \mathrm{M'Q.MQ} :: \mathrm{CN}^2 : \mathrm{CM}^2$$
$$:: \mathrm{NN'}^2 : \mathrm{MM'}^2.$$

Remark. All that has been demonstrated relative to the ordinates $M''Q$, $M'''Q$, may in like manner be shown to be true of the ordinates $M^{iv}Q'''$, $M^{v}Q'''$.

Cor. I. Tangents to the hyperbola at the extremities of a diameter, are parallel to each other; thus, the tangent at M' is parallel to the tangent at M.

Cor. II. Since the straight lines MN, $M'N'$ [Fig. 26] are parallel (Prop. XX. cor.), if they be produced to meet the tangents at M' and M, the quadrilateral .... $MN''M'N'''$ thus formed will be a parallelogram. This parallelogram is said to be constructed upon the diameter MM' and its conjugate.

PROPOSITION XXII.

THEOREM.

The parallelogram constructed upon any diameter and its conjugate, is equivalent to the rectangle of the axes.

That is [Fig. 26], the parallelogram $MN''M'N'''$ is equivalent to the rectangle $XX'X''X'''$.

Let MS and AS' be drawn perpendicular to CT; then the triangles MOS, AKS' are similar, and give

$$MO : AK :: MS : AS'.$$

But (Prop. XVII. cor. 3),

$$MO : AK :: CK : CO;$$

hence $$MS : AS' :: CK : CO,$$

and $$CK \times AS' = CO \times MS,$$

or $$2.(\text{the triangle } CAK) = 2.(\text{the triangle } CMO).$$

Now since CK = KX,

the triangle CAX = 2.(the triangle CAK);

and since MO = ON (Prop. XVIII. cor. 2),

the triangle CMN = 2.(the triangle CMO), and therefore the triangle CAX = the triangle CMN; and hence the rectangle CAXB = the parallelogram CNMT′, or the parallelogram MN″M′N‴=the rectangle XX′X″X‴.

Cor. I. If MO′ be drawn perpendicular to NN′, we have the parallelogram CNMT′ = CN×MO′;

and hence $$CN \times MO' = AC \times BC,$$

and $$CN^2 = \frac{BC^2 \times AC^2}{MO'^2}.$$

Cor. II. Draw FZ, F′Y [Fig. 25] perpendicular to the tangent at M; then since the tangent bisects the angle FMF′, the triangles FMZ, F′MY are similar to each other; they are also similar to the triangle MO′G. Hence we have

$$F'M : F'Y :: MG : MO',$$

and $$FM : FZ :: MG : MO';$$

and multiplying,

$$F'M \times FM : FZ \times F'Y :: MG^2 : MO'^2.$$

But (Prop. X.) $FZ \times F'Y = BC^2$,

and (Prop. XV. cor.) $MG = AC$;

hence $$F'M \times FM : BC^2 :: AC^2 : MO'^2,$$

and $$F'M \times FM = \frac{BC^2 \times AC^2}{MO'^2}$$

$$= CN^2. \qquad \text{(Cor. 1.)}$$

PROPOSITION XXIII.

THEOREM.

The difference of the squares of any diameter and its conjugate, is equal to the difference of the squares of the axes.

That is [Fig. 25],

$$MM'^2 - NN'^2 = AA'^2 - BB'^2.$$

From the triangles MCF and MCF′, we have (Geom. IV. 13 and 12), $MF^2 = MC^2 + CF^2 - 2\,CF \times CP$,

and $MF'^2 = MC^2 + CF'^2 + 2\,CF' \times CP$.

Adding, and recollecting that $CF = CF'$, we have

$$MF^2 + MF'^2 = 2\,MC^2 + 2\,CF^2;$$

and hence $2\,MC^2 = MF^2 + MF'^2 - 2\,CF^2$.

But (Prop. XXII. cor. 2), $2\,CN^2 = 2\,MF \times MF'$;

hence, subtracting,

$$\begin{aligned}
2\,MC^2 - 2\,CN^2 &= MF'^2 - 2\,MF.MF' + MF^2 - 2\,CF^2\\
&= (MF' - MF)^2 - 2\,CF^2\\
&= 4\,AC^2 - 2\,CF^2 \qquad \text{(Def. I.)}\\
&= 4\,AC^2 - 2\,AC^2 - 2\,BC^2\\
&\qquad\qquad \text{(since } CF^2 = AC^2 + BC^2)\\
&= 2\,AC^2 - 2\,BC^2; \text{ and consequently}\\
4\,MC^2 - 4\,CN^2 &= 4\,AC^2 - 4\,BC^2, \text{ or}\\
MM'^2 - NN'^2 &= AA'^2 - BB'^2.
\end{aligned}$$

CHAPTER IV.

IDENTITY OF THE PRECEDING CURVES WITH PLANE SECTIONS OF THE CONE.

DEFINITIONS.

I. If a straight line touching the circumference of a circle, and passing through a point without the plane of it, be moved around the circumference without ceasing to pass through the point, the surface generated is called a *conic surface*, and the solid terminated by the surface is called a *cone*.

II. The point is called the *vertex*, and the circle the *base* of the cone. The straight line drawn from the vertex to the centre of the circle, is called the *axis*. If the axis be perpendicular to the plane of the base, the cone is said to be *right*.

III. If the generating line be produced indefinitely in both directions, it is evident that the surface will extend indefinitely both above and below the vertex. The two parts of it separated at the vertex, are denominated the *upper* and *lower nappes* of the cone respectively.

Remark. It is evident that if a conic surface be cut by a plane passing through the vertex, the section will consist of two straight lines intersecting at the vertex.

PROPOSITION I.

THEOREM.

If a conic surface be cut by a plane parallel to the base, the section is the circumference of a circle having its centre in the axis.

Let ACBD [Fig. 29, Plate V.] be the base, S the vertex and SO the axis of a cone SACBD, and let A′C′B′D′ be a section made by a plane parallel to ACBD; then is A′C′B′D′ the circumference of a circle, and O′ its centre.

Let any two planes ASB, CSD be drawn through the axis, intersecting the planes ACBD, A′C′B′D′ in AB and CD, A′B′ and C′D′; then the triangles COS and C′O′S, AOS and A′O′S are similar, and give

$$CO : C'O' :: SO : SO',$$

and
$$AO : A'O' :: SO : SO';$$

and hence we have

$$CO : C'O' :: AO : A'O' :$$

but $CO = AO$; hence $C'O' = A'O'$.

That is, all the lines drawn from O′ to the section are equal, and consequently it is the circumference of a circle of which O′ is the centre.

COR. Any plane drawn through the axis of a cone cuts every section parallel to the base in its diameter.

PROPOSITION II.

THEOREM.

If a cone be cut by a plane meeting the conic surface on all sides, but not parallel to the base, the section is an ellipse.

Let IMI′m [Fig. 30] be a section of a conic surface by a plane meeting the surface on all sides, but not parallel to the base; then is IMI′m an ellipse.

Produce the cutting plane IMI′m till it meet the plane of the base in GH; and from the centre O, draw OG perpendicular to GH. Through OG and the vertex S, let a plane SAB be drawn, intersecting the surface in the lines SA, SB, and the cutting plane in II′ : then draw the planes FMEm, F′M′E′m′ parallel to the base, and intersecting the cutting plane in Mm, M′m′.

The lines AG, FE, F′E′ are parallel; so also are the lines GH, Mm, M′m′ (Geom. VI. 10) : but AG is perpendicular to GH; therefore Mm and M′m′ are perpendicular to FE and F′E′; and since FE and F′E′ are diameters of the circles FMEm and F′M′E′m′, we have (Geom. IV. 23, cor.)

$$MP^2 = EP \times FP,$$

and

$$M'P'^2 = E'P' \times F'P'.$$

But the line II′, the common section of the planes SBA and IMI′m, making with the lines EF and E′F′ and the sides of the cone SA and SB the similar triangles IEP and IE′P′, I′PF and I′P′F′, we have

$$EP : E'P' :: IP : IP',$$

and

$$FP : F'P' :: I'P : I'P';$$

hence, by multiplying these two proportions,

$$EP\times FP : E'P'\times F'P' :: IP\times I'P : IP'\times I'P';$$

and substituting for $EP\times FP$ and $E'P'\times F'P'$ their values MP^2 and $M'P'^2$, we have

$$MP^2 : M'P'^2 :: IP\times I'P : IP'\times I'P'.$$

That is, the squares of the ordinates to the diameter II′, are to each other as the products of the corresponding abscissas; and hence (Chap. I. Prop. 21, cor. 1) IMI′m is an ellipse, and II′ is one of its diameters.

Cor. If the angles SII′ and SAB are equal to each other, then the angle SII′ is equal to the angle I′FP, and the triangles IEP and I′FP are similar, and give

$$EP : IP :: I'P : FP,$$

or $$EP\times FP = IP\times I'P;$$

but $$EP\times FP = MP^2,$$

hence $$MP^2 = IP\times I'P.$$

Therefore, if MP be perpendicular to II′, the section IMI′m will be the circumference of a circle.

But in order that MP may be perpendicular to II′, the straight line GH must be perpendicular to GI′, and we shall then have GH perpendicular both to GI′ and GA; that is, the plane SAB must be perpendicular to the plane of the base, and also to the cutting plane. When all these circumstances meet, the cutting plane is said to be *anti-parallel* to the plane of the base.

PROPOSITION III.

THEOREM.

If a conic surface be cut by a plane parallel to another plane tangent to the surface, the section is a parabola.

Let M′Im′ [Fig. 31] be a section of a conic surface by a plane parallel to another plane tangent to the surface; then is M′Im′ a parabola.

Let the cutting plane meet the plane of the base in M′m′; from the centre O draw OP′ perpendicular to M′m′, and through S and OP′ let a plane SAB be drawn; then if NN′ be drawn tangent to the base at B, it will be parallel to M′m′ (Geom. III. 9), and the plane SNN′ will be the plane to which IM′m′ is parallel.

Draw the plane EMF parallel to the base, intersecting the cutting plane in Mm. Then since the intersections of parallel planes by a third plane are parallel, the quadrilateral PFBP′ is a parallelogram, and PF = P′B. But the similar triangles IPE and IP′A give

$$EP : AP' :: IP : IP';$$

and multiplying the first and second terms of this proportion respectively by PF and P′B, we have

$$EP \times PF : AP' \times P'B :: IP : IP'.$$

But since EF is a diameter of the circle EMF, and MP is perpendicular to EF, $EP \times PF = MP^2$:
similarly, $AP' \times P'B = M'P'^2$;
and substituting these values of EP.PF and AP′.P′B in the preceding proportion, we have

$$MP^2 : M'P'^2 :: IP : IP'.$$

That is, the squares of the ordinates are to each other as the corresponding abscissas; and hence (Chap. II. Prop. 11, cor 2) M′Im′ is a parabola, and IP′ is one of its diameters.

PROPOSITION IV.

THEOREM.

If both nappes of a conic surface be cut by a plane, the section is an hyperbola.

If M′Im′ [Fig. 32] be the section of a conic surface by a plane cutting both nappes, it is an hyperbola.

Let the figure be constructed as in the preceding proposition. Then the similar triangles EIP, AIP′ give

$$EP : AP' :: IP : IP';$$

and the similar triangles FI′P, BI′P′ give

$$FP : BP' :: I'P : I'P';$$

and multiplying these two proportions, we have

$$EP \times FP : AP' \times BP' :: IP \times I'P : IP' \times I'P'.$$

But $EP . FP = MP^2$, and $AP' . BP' = M'P'^2$; consequently,

$$MP^2 : M'P'^2 :: IP \times I'P : IP' \times I'P'.$$

Hence (Chap. III. Prop. IV. cor. 1) the section is an hyperbola, of which II′ is a diameter.

CHAPTER V.

AREA OF THE ELLIPSE AND PARABOLA.

PROPOSITION I.

THEOREM

The area of an ellipse is equal to the product of its semi-axes multiplied by the circumference of a circle whose diameter is unity.

The semi-transverse and semi-conjugate axes of an ellipse being designated by a and b respectively; the circumference of the circle whose diameter is unity, by $\pi = 3{,}1415926...$, (Geom. V. 14); and the area of the ellipse by s : then,

$$s = ab.\pi.$$

Let AA′ [Fig. 33] be the transverse axis of an ellipse ABA′B′. On AA′ as a diameter, describe the circumference of a circle ADA′D′, and let MM′ be a side of any inscribed polygon. From the vertices M, M′, ..., of the polygon, let ordinates MP, M′P′, ..., be drawn to the diameter AA′; and join the points N, N′, ..., in which these ordinates intersect the ellipse. A polygon will thus be inscribed in the ellipse, having NN′ for one of its sides.

Let CP $= x$, and CP′ $= x'$;
then PP′ $= x'-x$:
also let PM $= Y$, and P′M′ $= Y'$;
and PN $= y$, and P′N′ $= y'$.
Then (Geom. IV. 7) we shall have the

area of the trapezoid MM′P′P $= \frac{Y+Y'}{2}.(x'-x)$,

and of the trapezoid NN′P′P $= \frac{y+y'}{2}.(x'-x)$;

and hence, $\frac{\text{NN'P'P}}{\text{MM'P'P}} = \frac{y+y'}{Y+Y'}$.

But (Chap. I. Prop. VII.),

$$Y : y :: 2a : 2b, \quad \text{or } y = \frac{b}{a}Y,$$

and $$Y' : y' :: 2a : 2b, \quad \text{or } y' = \frac{b}{a}Y';$$

and hence $$y+y' = \frac{b}{a}(Y+Y'),$$

or $$\frac{y+y'}{Y+Y'} = \frac{b}{a};$$

therefore $$\frac{\text{NN'P'P}}{\text{MM'P'P}} = \frac{b}{a}.$$

In like manner, it may be shown that each of the trapezoids composing the polygon inscribed in the ellipse, is to the corresponding trapezoid of the polygon inscribed in the circle, in the ratio of b to a; from which we infer that the sum of all the first trapezoids, or the polygon inscribed in the ellipse, is to the polygon inscribed in the circle, in the same ratio. If then we denote these polygons by p and P, we shall have

$$\frac{p}{P} = \frac{b}{a}.$$

As this relation is true, whatever be the number of sides of the two polygons, it is true when that number is indefinitely increased; in which case the one polygon coincides with the ellipse, and the other with the circle. Representing then the surface of the ellipse by s, and that of the circle by S, we have

$$\frac{s}{S} = \frac{b}{a}.$$

Now π being the circumference of the circle whose diameter is 1, we have $2a\pi$ for the circumference of the circle whose diameter is $2a$, (Geom. V. 11); and therefore (Geom. V. 12) the area $S = a^2\pi$.

Hence, $s = \frac{b}{a}.S = \frac{b}{a}.a^2\pi = ab\pi.$

PROPOSITION II.

THEOREM.

The area of any semi-segment of a parabola is equal to two-thirds of the rectangle constructed on its extreme abscissa and ordinate.

Let AX [Fig. 34] be the axis of a parabola MAm. Construct the rectangle APMQ : then,

$$\text{AMP} = \tfrac{2}{3}\,\text{APMQ} = \tfrac{2}{3}\,xy.$$

On the arc AM assume a series of points M, M′, M″, ..., and through them draw perpendiculars and parallels to the axis AX : there will thus be formed a series of interior rectangles RPP′M′, R″P′P″M″, ...; and also a series of exterior rectangles R′QQ′M′, R‴Q′Q″M″, ... Then denoting by x, x', x'', ..., the abscissas, and by y, y', y'', ..., the ordinates of the points M, M′, M″, ...; and also by s the area of the interior rectangle RPP′M′, and by t the area of the corresponding exterior rectangle R′QQ′M′, we have

$$s = y'(x-x'),$$

and $$t = x'(y-y');$$

and hence, by division,

$$\frac{s}{t} = \frac{y'(x-x')}{x'(y-y')}.$$

But as M and M′ are points of the curve (Chap. II. Prop. II.),

$$y^2 = px, \quad \text{and } y'^2 = px';$$

and hence $x = \frac{y^2}{p}$, $x' = \frac{y'^2}{p}$, and $x-x' = \frac{y^2-y'^2}{p}$.

Substituting these values of x' and $x-x'$ in the above equation, we get

$$\frac{s}{t} = \frac{y'(y^2-y'^2)}{y'^2(y-y')} = \frac{y+y'}{y'} = 1+\frac{y}{y'}.$$

Denoting the areas of the remaining interior and exterior rectangles respectively by s', s'', ..., and t', t'', ..., and reasoning in a similar manner with regard to each of them, we shall have

$$\frac{s'}{t'} = 1+\frac{y'}{y''}, \qquad \frac{s''}{t''} = 1+\frac{y''}{y'''}, \qquad \&c.$$

But as the position of the points M, M′, M″, ..., is arbitrary, they may be so taken as to give the series of equal ratios

$$\frac{y}{y'} = \frac{y'}{y''} = \frac{y''}{y'''} = \ldots\ldots = 1+m,$$

(m being a constant fraction which we may assume as small as we please;) and this will be effected by taking on AY the lines

$$AQ' = AQ.\frac{1}{1+m}, \qquad AQ'' = AQ'.\frac{1}{1+m}, \qquad \ldots\ldots,$$

and drawing parallels to AX through the points Q′, Q″, ..., intersecting the curve in the required points M′, M″, ...

Hence, by substitution, the ratios $\frac{s}{t}$, $\frac{s'}{t'}$, ..., become

$$\frac{s}{t} = 2+m, \quad \frac{s'}{t'} = 2+m, \quad \frac{s''}{t''} = 2+m, \quad ......;$$

and we have (Geom. II. 10),

$$\frac{s+s'+s''+....}{t+t'+t''+....} = 2+m.$$

That is, the ratio of the sum of the interior to the sum of the exterior rectangles is equal to a constant quantity $2+m$.

Now it is evident that if we suppose m to diminish, the sum of the interior rectangles will approach to the semi-segment AMP, and the sum of the exterior rectangles to the area AMQ. If then we pass to the limit by making $m = 0$, we shall have

$$\frac{\mathrm{AMP}}{\mathrm{AMQ}} = 2,$$

or $$\mathrm{AMP} = 2\,\mathrm{AMQ};$$

and hence $$\mathrm{APMQ} = 3\,\mathrm{AMQ}:$$ consequently,

$$\mathrm{AMP} = 2\,\mathrm{AMQ} = \tfrac{2}{3}\,\mathrm{APMQ} = \tfrac{2}{3}\,xy.$$

CHAPTER VI.

CURVATURE OF THE ELLIPSE AND PARABOLA.

THE *curvature* of a curve, at any point, is its deflection from its tangent at that point.

Thus let AMB [Fig. 35] represent any curve whatever, and let TT′ be a tangent to the curve at the point M; then the curvature of AMB at M is its deflection from TT′ at that point.

The curvature at M is measured by the angle TMM′, formed by the tangent MT, and MM′ an arc of the curve so small that it may be considered a straight line.

Let the straight line MC′ be drawn perpendicular to TT′; and through M suppose a number of arcs of circles to be described, all having their centres in MC′, and some of them passing at the point M without the arc M′M″, others within it. Those which pass without the curve have evidently a less curvature than the curve has at M, and those which pass within the curve have a greater curvature. Now between these two systems of circles we may conceive a circle to be described, which shall lie neither within nor without the curve at the point M, but which shall coincide with an indefinitely small arc M′M″ of the curve at that point. This circle is called the *osculating circle* of the curve at the point M; and as it may be considered as having the same curvature that the curve itself has at that point, its radius is called the *radius of curvature* of the curve at the point M.

The length of the radius of curvature, at any point of a curve, indicates the degree of curvature at that point; for as the angle TMM′, or the measure of the curvature, increases, the radius of curvature diminishes, and conversely.

PROPOSITION I.

THEOREM.

The radius of curvature of an ellipse, at any point, is equal to the square of the semi-conjugate to the diameter passing through the point, divided by the perpendicular drawn from the centre of the ellipse to the tangent at the point.

At M any point of the ellipse ABA′B′ [Fig. 36] let the tangent TT′ be drawn. Draw the diameter MM‴, and NN′ conjugate to it; and from M′ infinitely near M, draw M′M″ parallel to the tangent. Draw also CR′ and MC′ perpendicular to the tangent, and let MC′ be the radius of curvature at the point M; then will

$$MC' = \frac{CN^2}{CR'}.$$

Since MM′ is an infinitely small arc of the ellipse, it will lie in the circumference described with the radius of curvature MC′; and hence MM′ may be considered as an arc of the circle of which MC′ is the radius. We therefore have (Geom. IV. 23, cor.)

$$M'O'^2 = MO'(2MC'-MO');$$

or, as MO′ may be neglected in comparison with 2MC′,

$$M'O'^2 = MO' \times 2MC';$$

and hence we have $MC' = \frac{M'O'^2}{2MO'}$.

or, as OO' is infinitely small compared with $M'O'$*,

$$MC' = \frac{M'O^2}{2\,MO'}.$$

But (Chap. I. Prop. XXI.)

$$M'O^2 : (2\,MC - MO)MO :: CN^2 : MC^2,$$

or, as MO is infinitely small compared with $2\,MC$,

$$M'O^2 : 2\,MC \times MO :: CN^2 : MC^2;$$

and hence $$M'O^2 = \frac{2\,MC \times MO \times CN^2}{MC^2}.$$

And the similar triangles MOO', MCR give

$$MO : MO' :: MC : MR,$$

and hence $$MO' = \frac{MO \times MR}{MC}.$$

Substituting the values of $M'O^2$ and MO' in the above expression for MC', we have

$$MC' = \frac{\dfrac{2\,MC \times MO \times CN^2}{MC^2}}{\dfrac{2\,MO \times MR}{MC}}$$

$$= \frac{CN^2}{MR}$$

$$= \frac{CN^2}{CR'}. \quad \text{..................} \quad [A]$$

---

* We have (Geom. IV. 23, cor.),

$$(MC' + C'O') : M'O' :: M'O' : O'M.$$

$M'O'$ is infinitely small with respect to $(MC' + C'O')$: hence MO' is infinitely small, compared with $M'O'$; that is, it is infinitely small, compared with an infinitely small quantity, or is *an infinitely small quantity of the second order.*

But, by similar triangles, we have

$$CR : RM :: OO' : O'M,$$

and CR and RM are both finite, and therefore quantities of the *same order;* consequently OO' and $O'M$ are of the same order, and therefore OO' is an infinitely small quantity of the second order. Hence OO' may be neglected in comparison with $M'O'$.

Cor. I. The radius of curvature at any point of an ellipse, is equal to the product of the squares of the semi-axes, divided by the cube of the perpendicular drawn from the centre of the ellipse to the tangent at the point.

From Chap. I. Prop. XXIII. cor. 1, we get

$$CN^2 = \frac{BC^2 \times AC^2}{MR^2};$$

and (Equa. [A]), $MC' = \frac{CN^2}{MR};$

hence by substitution we have

$$MC' = \frac{\frac{BC^2 \times AC^2}{MR^2}}{MR}$$

$$= \frac{BC^2 \times AC^2}{MR^3}$$

$$= \frac{BC^2 \times AC^2}{CR'^3}. \quad \text{.............} [B]$$

Cor. II. The radius of curvature at any point is equal to the semi-parameter, multiplied by the cube of the ratio of the radius vector of the point, to the perpendicular let fall from either focus upon the tangent at the point.

The triangles MF′Y, MGR are similar, because of the right angles at Y and R, and the alternate angles at M and F′, and give

$$MR : MG :: F'Y : F'M.$$

But (Chap. I. Prop. XVII.), $MG = AC$;

hence $MR : AC :: F'Y : F'M,$

and $MR = \frac{AC \times F'Y}{F'M}.$

This value of MR, substituted in Equation [B], gives

$$\mathrm{MC}' = \frac{\mathrm{BC}^2 \times \mathrm{AC}^2}{\dfrac{\mathrm{AC}^3 \times \mathrm{F'Y}^3}{\mathrm{F'M}^3}}$$

$$= \frac{\mathrm{BC}^2}{\mathrm{AC}} \times \frac{\mathrm{F'M}^3}{\mathrm{F\,Y}^3}$$

$$= \frac{p}{2} \cdot \left(\frac{\mathrm{F'M}}{\mathrm{F'Y}}\right)^3.$$

Cor. III. The radius of curvature at any point is equal to the cube of half the diameter which is conjugate to the diameter drawn through the point, divided by the rectangle of the axes.

From Chap. I. Prop. XXIII. cor. 1, we get

$$\mathrm{MR} = \frac{\mathrm{BC} \times \mathrm{AC}}{\mathrm{CN}};$$

hence by substitution in Equation [A], we have

$$\mathrm{MC}' = \frac{\mathrm{CN}^2}{\dfrac{\mathrm{BC} \times \mathrm{AC}}{\mathrm{CN}}} = \frac{\mathrm{CN}^3}{\mathrm{BC} \times \mathrm{AC}}.$$

Cor. IV. The radius of curvature at any point varies directly as the cube of the normal to the curve at that point.

Draw the ordinates MP, MP′; then the angles CT′T, MQT, being each the complement of the angle T′TC, are equal, and hence the triangles CR′T′, MPQ are similar, and

give $\mathrm{CR}' : \mathrm{CT}' :: \mathrm{MP} : \mathrm{MQ}$,

or $\mathrm{CR}' \times \mathrm{MQ} = \mathrm{MP} \times \mathrm{CT}'$,

or $\mathrm{MR} \times \mathrm{MQ} = \mathrm{CP}' \times \mathrm{CT}'$,

and $$\mathrm{MR} = \frac{\mathrm{CP}' \times \mathrm{CT}'}{\mathrm{MQ}}.$$

But (Chap. I. Prop. XVI. cor. 3), $CP' \times CT' = BC^2$;

hence $$MR = \frac{BC^2}{MQ},$$

and this value of MR, substituted in Equation [B], gives

$$MC' = \frac{BC^2 \times AC^2}{\frac{BC^6}{MQ^3}}$$

$$= \frac{AC^2}{BC^4} \times MQ^3;$$

but the ratio $\frac{AC^2}{BC^4}$ is constant for the same ellipse, and therefore MC′ varies as the cube of MQ.

PROPOSITION II.

THEOREM.

The radius of curvature of a parabola, at any point, is equal to twice the square of the radius vector drawn to the point, divided by the perpendicular let fall from the focus upon the tangent at the point.

At M any point of the parabola $M''AM^{iv}$ [Fig. 37], let the tangent TT′ be drawn. Draw the diameter DMX′; and from M′, infinitely near M, draw M′M″ parallel to TT′. From M and the focus F draw also MC′, FI, each perpendicular to TT′; and let MC′ be the radius of curvature. Then will

$$MC' = \frac{2\,FM^2}{FI}.$$

Since MM′ may be considered as an arc of the circle of which MC′ is the radius, we have, as in the preceding proposition,

$$\mathrm{MC'} = \frac{\mathrm{M'O'^2}}{2\,\mathrm{MO'}}. \quad \text{-------------} \quad [\mathrm{C}]$$

Now as O′P″ may be neglected in comparison with M′P″, we have $\mathrm{M'O'^2} = \mathrm{M'P''^2} = 4\,\mathrm{MD}\times\mathrm{MP''}$ (Ch. II. Prop. XI.)

$$= 4\,\mathrm{FM}\times\mathrm{MP''}.$$

Also the triangles MO′O″, MFI, being right-angled at O′ and I, and having the angles at F and M equal, are similar, and give

$$\mathrm{MO''} : \mathrm{MO'} :: \mathrm{FM} : \mathrm{FI};$$

and hence $$\mathrm{MO'} = \frac{\mathrm{FI}\times\mathrm{MO''}}{\mathrm{FM}}.$$

But since the angles MO″P″, MP″O″ are equal respectively to the angles TMO″, T′MP″, and TMO″ is equal to T′MP″ (Chap. II. Prop. IV. cor. 3), we have

$$\mathrm{MO''P''} = \mathrm{MP''O''},$$

and hence $$\mathrm{MO''} = \mathrm{MP''};$$

consequently $$\mathrm{MO'} = \frac{\mathrm{FI}\times\mathrm{MP''}}{\mathrm{FM}}.$$

Substituting these values of M′O′ and MO′ in Equa. [C], we get

$$\mathrm{MC'} = \frac{4\,\mathrm{FM}\times\mathrm{MP''}}{\dfrac{2\,\mathrm{FI}\times\mathrm{MP''}}{\mathrm{FM}}}$$

$$= \frac{2\,\mathrm{FM}^2}{\mathrm{FI}}.$$

Cor. The radius of curvature of the parabola is equal to the semi-parameter, multiplied by the cube of the ratio of the radius vector of the point, to the perpendicular drawn from the focus to the tangent at the point.

Since $\mathrm{FA} \times \mathrm{FM} = \mathrm{FI}^2$, (Chap. II. Prop. V. cor.)

or $$\frac{\mathrm{FA} \times \mathrm{FM}}{\mathrm{FI}^2} = 1, \text{ we have}$$

$$\begin{aligned}\mathrm{MC}' &= \frac{2\,\mathrm{FM}^2}{\mathrm{FI}} \times \frac{\mathrm{FA} \times \mathrm{FM}}{\mathrm{FI}^2} \\ &= 2\,\mathrm{FA} \times \left(\frac{\mathrm{FM}}{\mathrm{FI}}\right)^3 \\ &= \frac{p}{2} \cdot \left(\frac{\mathrm{FM}}{\mathrm{FI}}\right)^3.\end{aligned}$$

THE END

Fig. 1

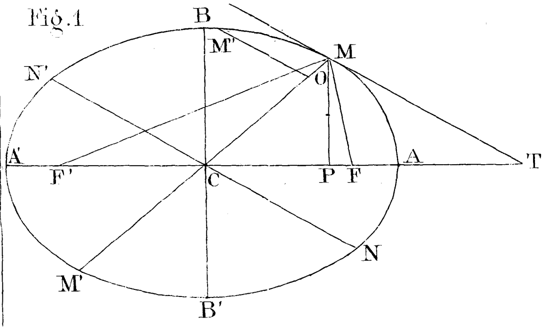


Fig. 2

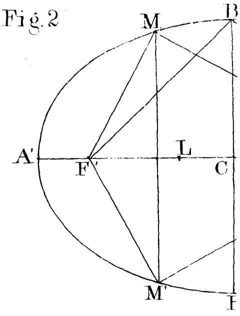


Fig. 4

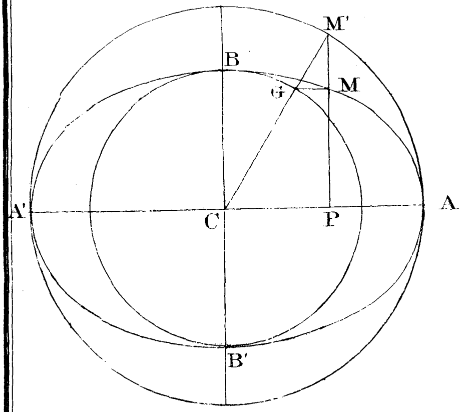


Fig. 5

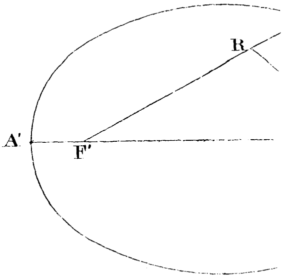


Fig. 8

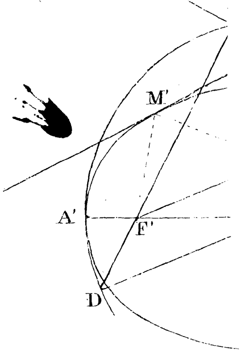


Fig. 9

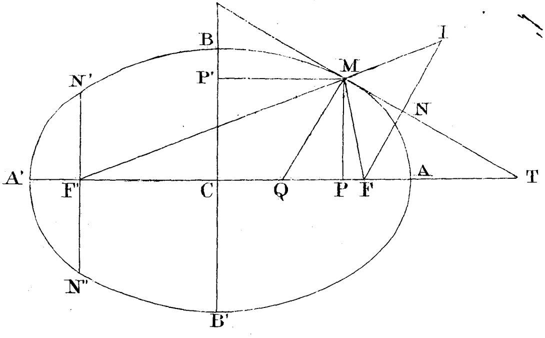


Fig. 10

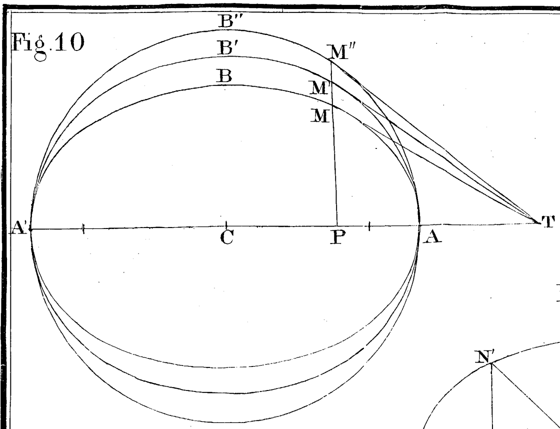


Fig. 11

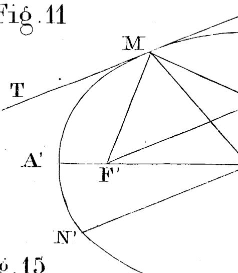


Fig. 15

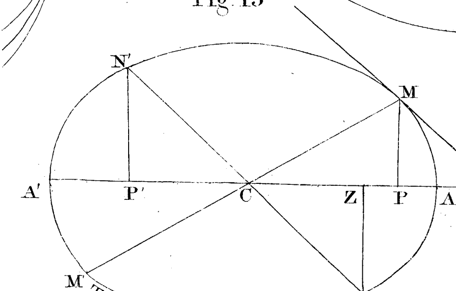
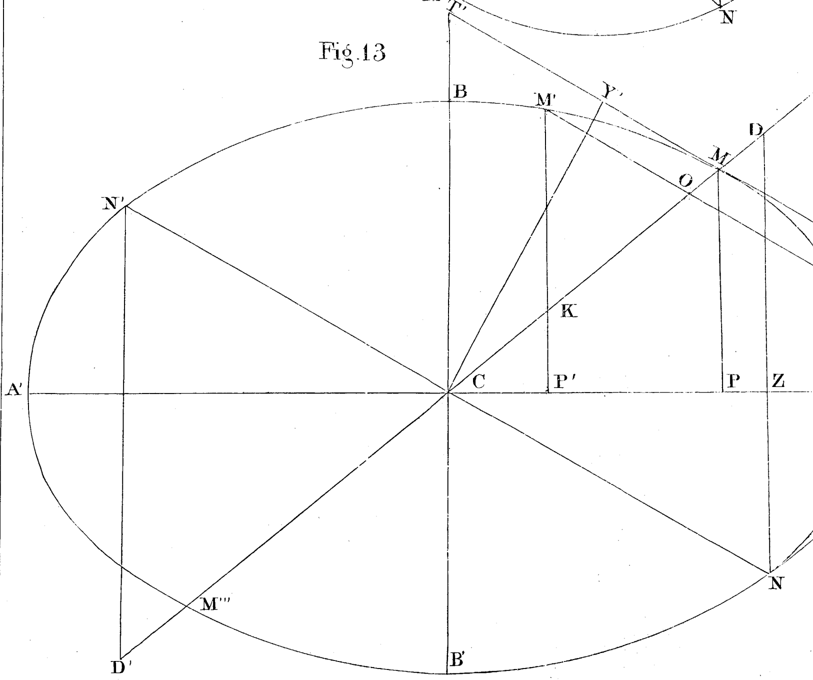


Fig. 13



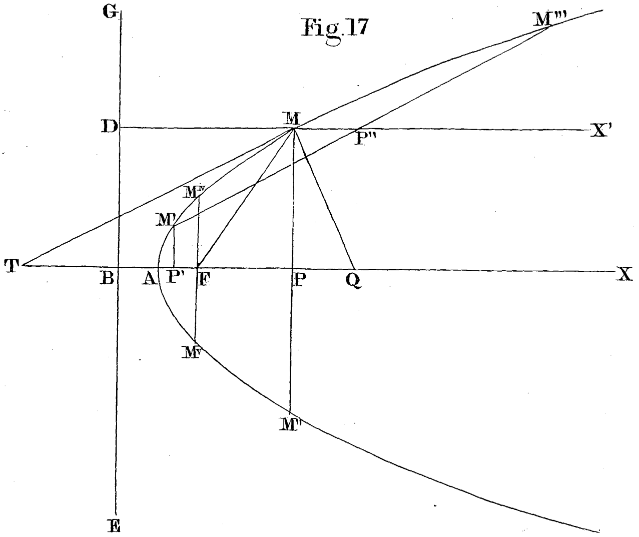
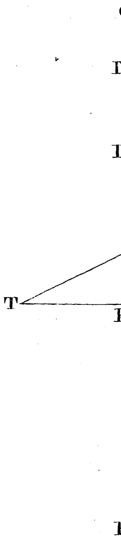


Fig. 17



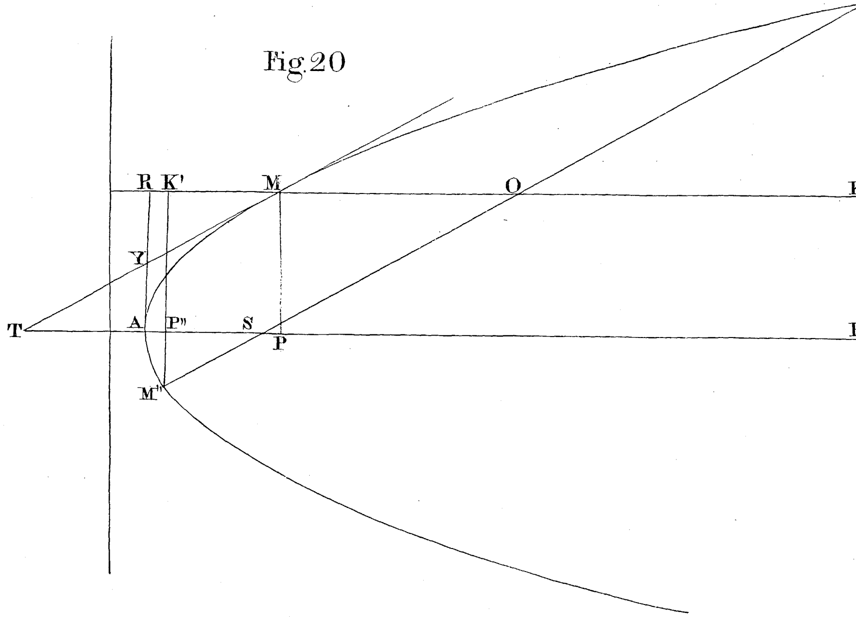


Fig. 20

Fig. 21

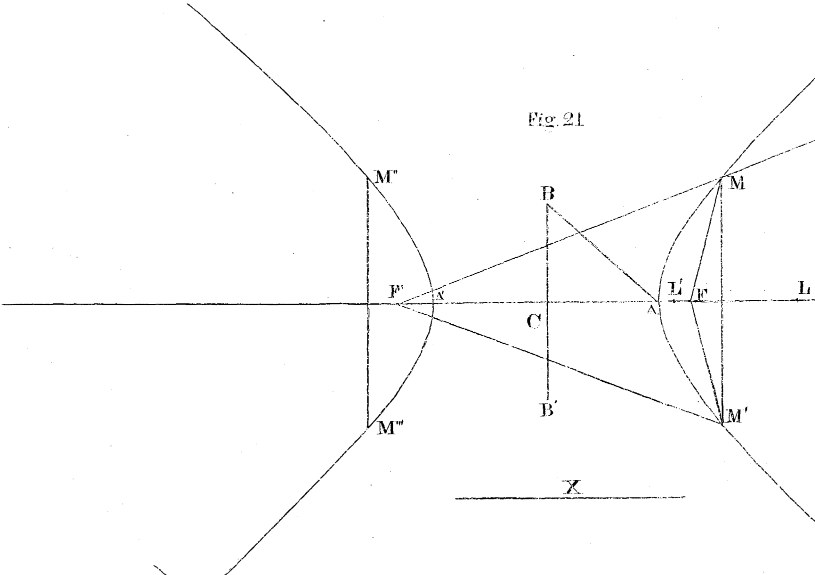
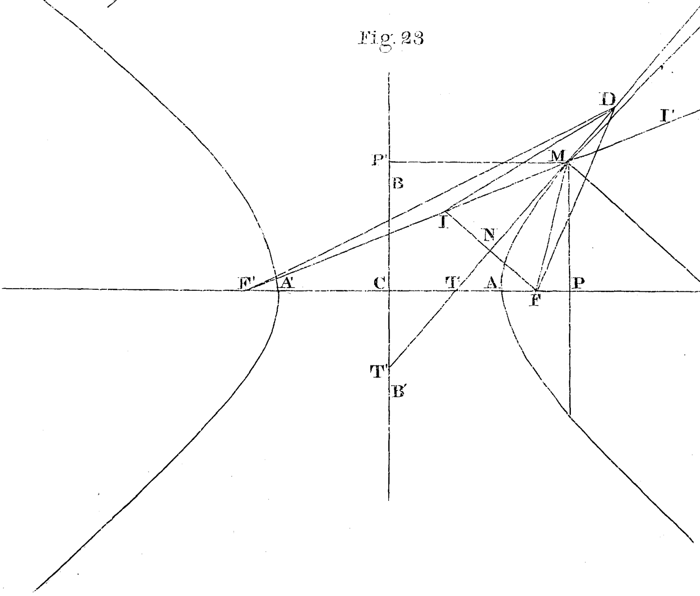


Fig. 23



The Student is requested to construct Fi

Fig. 25

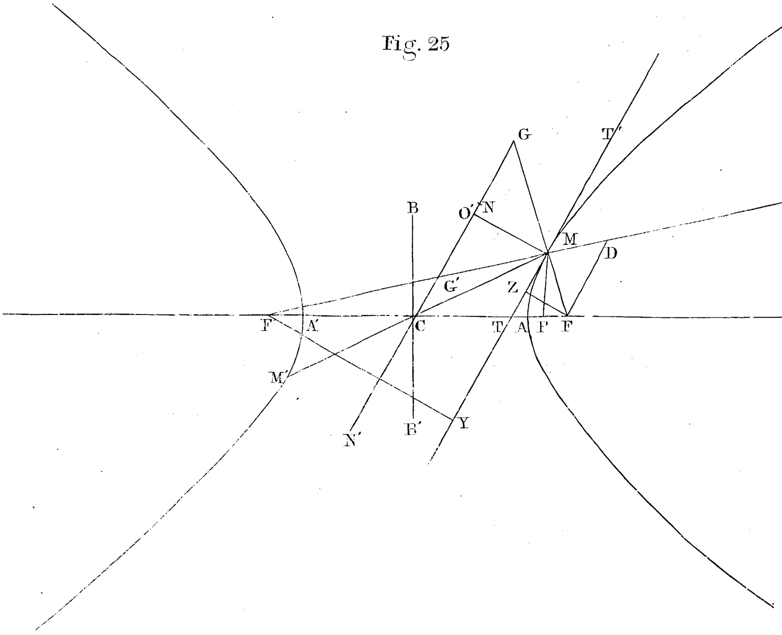


Fig. 29

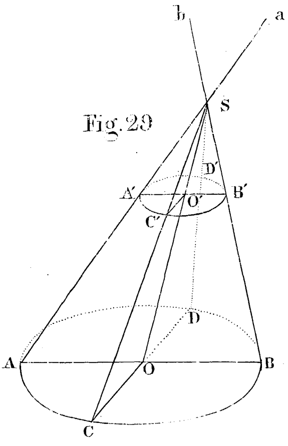


Fig. 30